REMARKS ON FIXED POINT THEOREMS OF RICCERI

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1. Introduction

In [R1], B. Ricceri considered spaces admitting a continuous bijection onto $[0,1]$ (simply, we will call them $[0,1]$-*spaces*) and, based on a new alternative principle for multimaps involving $[0,1]$-spaces, obtained new mini-max theorems in full generality and transparence. Several further consequences of the principle have been investigated in successive works; see [R2, R3, C, CB].

In the present paper, we deduce some fixed point theorems for $[0,1]$-spaces from Ricceri's alternative principle. Even though these theorems are consequences of known theorems for an interval $[a,b]$, in general, they seem to be quite new.

After giving necessary results of Ricceri [R1] in Section 2, we introduce new fixed point theorems of Ricceri for $[0,1]$-spaces in Section 3. In the last section, we discuss some known general fixed point theorems in order to compare with the new theorems of Ricceri.

2. Ricceri's alternative principle

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x,y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

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A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c) if for each closed set $B \subset Y$, the set $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^{-}(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map T with closed values is closed.

Throughout this paper, all topological spaces are assumed to be Hausdorff, and t.v.s. means topological vector spaces.

The following is the alternative principle of Ricceri [R1, Theorem 2.1]:

Theorem A. *Let X be a connected topological space, and Y a $[0,1]$-space. Let $F, G : X \multimap Y$ be maps satisfying one of the following two conditions:*

(i) *F, G are l.s.c. with connected values;*

(ii) *F, G are u.s.c. with compact connected values.*

Under such assumptions, at least one of the following two assertions does hold:

(a) *$F(X) \neq Y$ and $G(X) \neq Y$.*

(b) *There exists some $\tilde{x} \in X$ such that $F(\tilde{x}) \cap G(\tilde{x}) \neq \emptyset$.*

Szabó [S] defined that a map $F : X \multimap [0,1]$ is continuous whenever $F(x) = [f_0(x), f_1(x)]$ for $x \in X$, where $f_0, f_1 : X \to [0,1]$ are continuous. Then, he obtained the following trivial consequence of Theorem A.

Corollary. *Let X be a connected space, and $F, G : X \multimap [0,1]$ continuous maps such that $F(X) = [0,1]$. Then there exists $x_0 \in X$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.*

From Theorem A, Ricceri obtained the following consequence [R1, Theorem 2.2]:

Theorem B. *Let X be a topological space, Y a $[0,1]$-space, and S a connected subset of $X \times Y$. Moreover, let $\Phi : X \multimap Y$ be a multimap which is either l.s.c. with connected values, or u.s.c. with compact connected values. Then, at least one of the following holds:*

(a_1) *$p_Y(S) \neq Y$ and $\Phi(p_X(S)) \neq Y$, where p_X and p_Y are projections from $X \times Y$ to X and Y, resp.*

(a_2) *There exists some $(\tilde{x}, \tilde{y}) \in S$ such that $\tilde{y} \in \Phi(\tilde{x})$.*

3. Ricceri's fixed point theorems

In this section, we deduce fixed point theorems on multimaps defined on connected $[0,1]$-spaces.

Theorems. *Let X be a connected $[0,1]$-space. Then a map $F : X \multimap X$ has a fixed point if it satisfies one of the following conditions:*

(I) *F is l.s.c. with connected values.*
(II) *F is u.s.c. with compact connected values.*
(III) *F has connected graph.*
(IV) *$F(x)$ is connected and $F^{-1}(y)$ is open for each $x, y \in X$.*
(V) *F is a closed compact map with connected values.*

Proof. Suppose that F has no fixed point. Let φ be a continuous bijection from X onto $[0,1]$ and $H : X \multimap [0,1] \times [0,1]$ defined by

$$H(x) = \varphi(F(x)) \times \{\varphi(x)\} \quad \text{for} \quad x \in X.$$

We claim that the set $H(X)$ is connected:

(I) Since $x \mapsto \varphi(F(x))$ is l.s.c. with connected values, so is H. Hence $H(X)$ is connected by [Hi, Theorem 3.1].

(II) Since $x \mapsto \varphi(F(x))$ is u.s.c. with compact connected values, so is H by [Br]. Hence $H(X)$ is connected by [Hi, Theorem 3.1].

(III) Since F has connected graph, $F(X)$ and $\varphi(F(X))$ are connected. Hence $H(X) = \varphi(F(X)) \times \{\varphi(X)\} = \varphi(F(X)) \times [0,1]$ is connected.

Let

$$A = \{(s,t) \in [0,1] \times [0,1] : s < t\} \quad \text{and} \quad B = \{(s,t) \in [0,1] \times [0,1] : s > t\}.$$

Since F has no fixed point, we have

$$H(X) \subset A \cup B.$$

Let $x_0 = \varphi^{-1}(0)$ and $x_1 = \varphi^{-1}(1)$. Choose $y_0 \in F(x_0)$ and $y_1 \in F(x_1)$. Then $\varphi(y_0) > 0$, since $\varphi(y_0) = 0$ would imply $x_0 = y_0 \in F(x_0)$. Likewise, we have $\varphi(y_1) < 1$. Consequently,

$$(0, \varphi(y_0)) \in A \cap H(X) \quad \text{and} \quad (1, \varphi(y_1)) \in B \cap H(X).$$

Since A, B are open and disjoint, this contradicts the connectedness of $H(X)$. This completes the proofs of (I)-(III).

(IV) Since $F^{-1}(y)$ is open for each $y \in X$, F is l.s.c. Indeed, for each open set $\Omega \subset X$, we have

$$F^{-1}(\Omega) = \{x \in X : F(x) \cap \Omega \neq \emptyset\} = \bigcup_{y \in \Omega} F^{-1}(y)$$

is open. Therefore, (IV) follows from (I).

(V) It is well-known that a closed compact map is u.s.c. with compact values. Therefore, (V) follows from (II).

Remark 1. Theorems I-V are all simple consequences of Ricceri [R1] as follows:

(I) Theorem A(i) with $X = Y$ and $G = 1_X$.

(II) Theorem A(ii) with $X = Y$ and $G = 1_X$.

(III) Theorem B with $X = Y$, $S = F$, and $\Phi = 1_X$.

(IV) [R1, Theorem 2.3(α)] with $X = Y$, $T = F$, and $S = 1_X$.

(V) [R1, Theorem 2.3(β)] with $X = Y$, $T = F$, and $S = 1_X$.

However, for reader's convenience, we give their proofs as above based on Ricceri's argument in the proof of Theorem A.

Remark 2. As is noted by Ricceri [R1, Remark 2.1], in the proof of Theorems, it is clear that Theorems are still true if $[0, 1]$ is replaced by any space Z having the following property: there are two open (or closed) subsets A, B of $Z \times Z$ and two points $s_0, t_0 \in Z$ such that $(Z \times Z) \backslash \Delta \subset A \cup B$, $A \cap B \subset \Delta$, $\{s_0\} \times (Z \backslash \{s_0\}) \subset A$, and $\{t_0\} \times (Z \backslash \{t_0\}) \subset B$, where Δ is the diagonal of $Z \times Z$.

In [S], Szabó raised the problem: how can we generalize his Corollary (to Theorem A) for other spaces instead of $[0,1]$. The aforementioned type of a space Z is a quite general solution to the problem.

Remark 3. Theorems I-V work for any bounded closed interval $X = [a,b]$ or for

$$X = \{(0,0)\} \cup \{(x,y) : x \in (0,1] \quad \text{and} \quad y = \sin \frac{1}{x}\} \subset \mathbf{R}^2.$$

The *connectedness of* X in Theorems I-V is essential: for example, for the $[0,1]$-space X given by

$$X = \{-1\} \cup (0,1) \cup \{2\} \subset \mathbf{R},$$

we give counterexamples $F = f : X \to X$ violating (I)–(V) as follows:

(I),(II),(V) $f(-1) = 2$, $f(2) = -1$, and $f(x) = \sqrt{x}$ for $x \in (0,1)$.

(III) There is no $F : X \multimap X$ having connected graph.

(IV) $f(-1) = 2$, $f(2) = -1$, and $f(x) = -1$ for $x \in (0,1)$.

Even if X is connected, the *connectedness of the graph* in Theorem III is essential: for example, for $X = [0,1]$, let

$$f(x) = 1 \text{ if } x \in [0,1/2] \text{ and } f(x) = 0 \text{ if } x \in (1/2,1].$$

4. Comparisons with known results

For $X = [0,1]$, since connectedness of a subset is same to convexity, some of Theorems I–V are consequences of known results. However, Theorems I and III seem to be new even for $X = [0,1]$. On the other hand, for any $[0,1]$-spaces; for example,

$$X = \{(0,0)\} \cup \{(x,y) : x \in (0,1] \quad \text{and} \quad y = \sin \frac{1}{x}\} \subset \mathbf{R}^2,$$

Theorems I–V seem to be quite new.

For convex spaces (simply, convex subsets of a t.v.s.) or some other spaces, the following sequence of implications for nonempty subsets is clear:

$$\text{convex} \Longrightarrow \text{star-shaped} \Longrightarrow \text{contractible} \Longrightarrow \text{acyclic} \Longrightarrow \text{connected.}$$

Here, a space is *contractible* if the identity map is homotopic to a constant map; and a nonempty space is *acyclic* if it is connected and its Čech homology (with a fixed coefficient field) is zero in dimensions greater than zero.

In this section, we collect generalized forms of known fixed point theorems which are similar to Theorems I–V for $X = [0, 1]$.

The following is a particular form of Reich [Re, Theorem 1.9]:

Theorem I′. *Let X be a nonempty compact convex subset in a locally convex completely metrizable t.v.s. E. Let $F : X \multimap X$ be a l.s.c. map with closed convex values. Then F has a fixed point.*

For $X = [0, 1]$, Theorem I does not assume the closedness of values of F. Therefore, Theorems I and I′ are not comparable.

For the definition of a *lc space*, see [B, V, T]. Note that an ANR (metric) is an lc space and a finite union of compact convex subsets of a locally convex t.v.s. is an lc space.

The following is due to Begle [B]:

Theorem II′. *Let X be a compact acyclic lc space and $F : X \multimap X$ a closed map with acyclic values. Then F has a fixed point.*

For $X = [0, 1]$, Theorem II′ extends Theorem II. However, for a connected $[0, 1]$-space X, Theorems II and II′ seem to be not comparable.

For topological spaces X and Y, $F : X \multimap Y$ is called a *connectivity map* if the graph over each connected subset of X is a connected set. This is introduced by Nash for single-valued case.

The following is due to Girolo [G, Corollary 2]:

Theorem III′. *Let $f : X \to X$ be a connectivity function and X a compact convex subset of a normed vector space. Then f has a fixed point.*

Even for $X = [0,1]$, Theorem III is general than Theorem III′. In fact, for a connected space X, a connectivity map $F : X \multimap X$ has connected graph, but not conversely; for example, $F : [-1,1] \multimap [-1,1]$ is given by $F(x) = \{y \in [-1,1] : y^2 = (x+1)/2\}$ for $x \in [-1,1]$.

The following is known [Ho, Théorème 2]:

Theorem IV′. *Let X be a compact contractible space and $F : X \multimap X$ a map such that*

(a) *for each open set O in X, the set $\bigcap_{x \in O} F(x)$ is empty or contractible; and*
(b) *for each $y \in X$, $F^{-1}(y)$ is open.*

Then F has a fixed point.

For more general results than Theorem IV′, see Park *et al.* [PJ, PK].

The following particular case of Theorem IV′ is known as the Fan-Browder fixed point theorem:

Corollary IV′. *Let X be a compact convex space and $F : X \multimap X$ a map such that*

(a) *for each $x \in X$, $F(x)$ is convex; and*
(b) *for each $y \in X$, $F^{-1}(y)$ is open.*

Then F has a fixed point.

For $X = [0,1]$, Theorem IV follows from Corollary IV′. On the other hand, for any connected $[0,1]$-space X, Theorem IV and Corollary IV′ seem to be not comparable.

There are a lot of generalizations of Corollary IV′; see Park [P1] and Park and Kim [PK].

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset A of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : A \to X$ such that $x - h(x) \in V$ for all $x \in A$ and $h(A)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. The spaces l^p, L^p, and the Hardy spaces H^p for $0 < p < 1$ are admissible. For other examples, see Hadžić [H], Weber [W], and references therein.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in finite dimensional subspace of E. We define the *"better" admissible class* $\mathfrak{B}$ of multimaps defined on X as follows:

$F \in \mathfrak{B}(X,Y) \Longleftrightarrow F : X \multimap Y$ is a map such that for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (u.s.c. with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (u.s.c. with R_δ values), the acyclic maps $\mathbb{V}$ (u.s.c. with acyclic values), the Powers maps $\mathbb{V}_c$ (finite composites of acyclic maps), the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of t.v.s.), admissible maps of Górniewicz, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park *et al.*, approximable maps of Ben-El-Mechaiekh and Idizk, and many others. Those subclasses are all examples of the admissible class $\mathfrak{A}_c^\kappa$. See Park [P2, P4].

The following new fixed point theorem is due to the author [P4] recently.

Theorem V′. *Let E be a t.v.s. and X an admissible convex subset of E. Then any closed compact map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

The following is due to the author [P3] earlier.

Corollary V′. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact acyclic map $F \in \mathbb{V}(X, X)$ has a fixed point.*

For $X = [0, 1]$, Corollary V′ is same to Theorem V.

Note that Theorem V′ (and Corollary V′) extends and unifies a large number of known or new fixed point theorems.

Finally, in order to satisfy questions raised by an inconsiderate reviewer of [PJ] in MR 96m:47109, we give examples of Theorem IV′ and $\mathfrak{B}([0, 1], [0, 1])$.

Example 1. Let X be the compact contractible space given by

$$X = ([0, 1] \times \{0\}) \cup (\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\} \times [0, 1]) \subset \mathbf{R}^2.$$

Let $F : X \multimap X$ be defined by

$$F(x, y) = \{(x', y') \in X : 0 \le y' < \frac{1}{2}\} \quad \text{for} \quad (x, y) \in X.$$

Then each value is contractible and clearly condition (a) of Theorem IV′ is satisfied. On the other hand, $F^{-1}(x', y') = X$ if $y' < 1/2$ and $F^{-1}(x', y') = \emptyset$ if $y' \ge 1/2$. Therefore condition (b) also holds.

Let $G : X \multimap X$ be defined by

$$G(x, y) = \begin{cases} \{(x', y') \in X : x' = 1\} & \text{if} \quad x = 0 \\ \{(x', y') \in X : x' < x\} & \text{if} \quad x > 0. \end{cases}$$

Then G does not satisfy condition (a) of Theorem IV′.

Let $H : X \multimap X$ be defined by

$$H(x, y) = \begin{cases} \{(x', y') \in X : x < x'\} & \text{if} \quad x < 1 \\ \{(x', y') \in X : x' = 0\} & \text{if} \quad x = 1. \end{cases}$$

Note that

$$H^{-1}(x', y') = \{(x, y) \in X : x = 1\} \quad \text{if} \quad x' = 0,$$

which is not open, and H does not satisfy condition (b) of Theorem IV′.

Note that G and H are fixed point free.

Example 2. We give examples of maps in $\mathfrak{B}([0,1],[0,1])$. Let $X = [0,1]$ or $[a,b]$ and P be a closed subinterval of X.

(1) Let $F : X \multimap X$ be a map such that $F(x)$ is connected and $F^{-1}(y)$ is open for $x, y \in X$. Then for any continuous map $f : F(P) \to P$ and each $x, y \in P$, $fF(x)$ is connected and $(fF)^{-1}(y) = (F^{-1}f^{-1})(y)$ is open as a union of open sets. Therefore, $f(F|_P)$ has a fixed point by Theorem IV.

(2) Let $F : X \to X$ be a single-valued connectivity function. Then so is $f(F|_P)$ is also a connectivity function and has a fixed point by Theorem III′.

(3) Let $F : X \multimap X$ be a closed map with connected values. Then for any continuous map $f : F(P) \to P$, $f(F|_P)$ is also a closed map with connected values and hence has a fixed point by Theorem V.

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