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COINCIDENCE THEOREMS FOR THE BETTER ADMISSIBLE MULTIMAPS AND THEIR APPLICATIONS

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1. INTRODUCTION

Recently, in a sequence of papers [1-18], the author introduced the admissible classes $\mathfrak{A}_c^\kappa$ of multimaps, which are large enough to include most of multimaps appearing in nonlinear analysis and algebraic topology. For the admissible classes, we established the foundations of the KKM theory via coincidences of such multimaps [2,3] and the fixed point theory in topological vector spaces [1,9].

Some of those new results were, consecutively, applied to the following topics:

(1) Best approximation problems [4,6,11,14].

(2) Generalized variational or quasi-variational inequalities and generalized complementarity problems [17-22].

(3) Generalized Leray-Schauder or Birkhoff-Kellogg theorems [5,7,12,13,16].

(4) Extensions to generalized convex spaces [23-26].

(5) Applications of a generalized minimax inequality [18].

(6) Openess of multifunctions [15].

On the other hand, more recently, Chang and Yen [27] extended the classes $\mathfrak{A}_c^\kappa$ to multimaps having the KKM property and obtained some generalized results in the KKM theory and fixed point theory. We will denote their class by $\mathfrak{K}$.

Let X be a convex space and Y a Hausdorff space. In this paper, we define a new "better" admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P)$ has a fixed point.

Our new class contains the admissible class $\mathfrak{A}_c^\kappa$ due to the author and closed maps in $\mathfrak{K}$ due to Chang and Yen [27].

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In Section 2, we obtain a basic coincidence theorem for the class $\mathfrak{B}$. Section 3 deals with a matching theorem and KKM theorem, which are basis of the KKM theory and have many applications. In Sections 4 and 5, we deduce fixed point theorems for compact or condensing multimaps in $\mathfrak{B}$ or in some related classes of multimaps.

2. TWO CLASSES $\mathfrak{A}_c^\kappa$ AND $\mathfrak{K}$

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y having nonempty values. Note that $y \in Tx$ is equivalent to $x \in T^- y$ and, for $B \subset Y$, let $T^-(B) := \{x \in X : Tx \cap B \neq \emptyset\}$ and $T^+(B) := \{x \in X : Tx \subset B\}$. For topological spaces X and Y, a map $T : X \multimap Y$ is *upper semicontinuous* if $T^+(B)$ is open for each open set B in Y; *lower semicontinuous* if $T^+(B)$ is closed for each closed set B in Y; and *continuous* if it is upper and lower semicontinuous.

Let X be a set (in a vector space), D a nonempty subset of X, and $\langle D\rangle$ the set of all nonempty finite subsets of D. Then (X, D) is called a *convex space* if the convex hull $\operatorname{co} N$ of any $N \in \langle D\rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. Such convex hulls are called *polytopes*. A subset A of X is said to be *D-convex* if, for any $N \in \langle D\rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [28].

For a convex space (X, D), a map $G : D \multimap X$ is called a *KKM map* if $\operatorname{co} N \subset G(N)$ for each $N \in \langle D\rangle$. If G is a closed-valued *KKM map*, it is well-known that $\{Gx\}_{x\in D}$ has the finite intersection property. The *KKM theory* is the study of KKM maps and their applications. For the literature, see Park [3,9,10].

Let X and Y be topological spaces. An *admissible class* $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is one such that, for each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma x \subset Tx$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact-valued; and

(iii) for each polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers map $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of topological vector spaces), admissible maps of Górniewicz, permissible maps of Dzedzej, and others. Further, $\mathbb{K}_c^+$ due to Lassonde [29], $\mathbb{V}_c^+$ due to Park *et al.* [30], and approximable maps $\mathbb{A}^\kappa$ due to Ben-El-Mechaiekh and Idzik [31] are examples of $\mathfrak{A}_c^\kappa$.

Let (X, D) be a convex space, Y a Hausdorff space, and $T : X \multimap Y$. We say that T has *the KKM property* provided that the family $\{Sx : x \in D\}$ has the finite intersection property whenever $S : D \multimap Y$ has closed values and $T(\operatorname{co} N) \subset S(N)$ for each $N \in \langle D\rangle$. See [27].

$T \in \mathfrak{K}(X, Y) \iff T : X \multimap Y$ has the KKM property.

Chang and Yen [27] obtained the following:

(a) Let X be a convex subset of a locally convex t.v.s. and $T \in \mathfrak{K}(X,X)$ a compact closed map. Then T has a fixed point.

(b) $T \in \mathfrak{K}(X,Y)$ if and only if $T|_P \in \mathfrak{K}(P,Y)$ for each polytope P in X.

(c) If $T \in \mathfrak{K}(X,Y)$ and $f \in \mathbb{C}(Y,Z)$, where Z is a Hausdorff space, then $fT \in \mathfrak{K}(X,Z)$.

(d) If Y is normal and P is a polytope in X, and if $T : P \multimap Y$ is a map such that for each $f \in \mathbb{C}(Y,P)$, fT has a fixed point in P, then $T \in \mathfrak{K}(P,Y)$.

(e) $\mathfrak{A}_c^\kappa(X,Y) \subset \mathfrak{K}(X,Y)$. This is due to Park [3, Corollary 2].

Throughout this paper, t.v.s. means Hausdorff topological vector spaces.

3. THE CLASS $\mathfrak{B}$ AND COINCIDENCE THEOREMS

We now define a new "better" admissible class defined on a convex space (X,D):

$F \in \mathfrak{B}(X,Y) \iff$ for any polytope P in X and any $f \in \mathbb{C}(F(P),P)$, $f(F|_P) : P \multimap P$ has a fixed point.

For the simplicity, we assume that Y is a Hausdorff space.

PROPOSITION. *We have the following :*

(i) $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$.

(ii) *If $F \in \mathfrak{K}(X,Y)$ is closed, then $F \in \mathfrak{B}(X,Y)$. But not conversely.*

(iii) *Any map $F : X \multimap Y$ such that $F|_P : P \multimap Y$ is approachable for each polytope P in X belongs to $\mathfrak{B}(X,Y)$.*

(iv) *$\mathfrak{B}(X,Y) \subset \mathfrak{K}(X,Y)$ whenever Y is normal.*

(v) *In the class of closed compact maps, two subclasses $\mathfrak{B}$ and $\mathfrak{K}$ coincide.*

Proof. (i) Clear by the definition of $\mathfrak{A}_c^\kappa$.

(ii) This follows from the properties (a), (b) and (c). For the converse, for example, a Fan-Browder type map $F \in \mathfrak{K}(X,Y)$ with open values and nonempty convex fibers is not closed in general, but belongs to $\mathfrak{B}(X,Y)$

(iii) See [32].

(iv) This follows from (d) and (b).

(v) This follows from (ii) and (iv).

Remark. However, $\mathfrak{K}$ and $\mathfrak{B}$ may not be comparable in general.

Now, we have the following coincidence theorem:

THEOREM 1. *Let (X,D) be a convex space, Y a Hausdorff space, and $F,G : X \multimap Y$ maps satisfying*

(1) *$F \in \mathfrak{B}(X,Y)$ is compact;*

(2) *for each* $y \in F(X)$, $G^{-}y$ *is* D-*convex; and*

(3) $\{\operatorname{Int} Gx : x \in D\}$ *covers* $\overline{F(X)}$.

Then F *and* G *have a coincidence point* $x_0 \in X$*; that is,* $Fx_0 \cap Gx_0 \neq \emptyset$.

Proof. Since $\overline{F(X)}$ is compact and included in $\bigcup\{\operatorname{Int} Gx : x \in D\}$, there exists an $N = \{x_1, x_2, \ldots, x_n\} \in \langle D \rangle$ such that $\overline{F(X)} \subset \bigcup\{\operatorname{Int} Gx : x \in N\}$. Let $\{\lambda_i\}_{i=1}^n$ be the partition of unity subordinated to this cover of the Hausdorff compact space $\overline{F(X)}$, and $P := \operatorname{co} N \subset X$. Define $f : \overline{F(X)} \to P$ by

$$fy = \sum_{i=1}^{n} \lambda_i(y) x_i = \sum_{i \in N_y} \lambda_i(y) x_i$$

for $y \in \overline{F(X)} \subset Y$, where

$$i \in N_y \Longleftrightarrow \lambda_i(y) \neq 0 \Longrightarrow y \in \operatorname{Int} Gx_i \subset Gx_i.$$

Then $x_i \in G^{-}y$ for each $i \in N_y$. Clearly f is continuous and, by (2), we have $fy \in \operatorname{co}\{x_i : i \in N_y\} \subset G^{-}y$ for each $y \in F(X)$. Since P is a polytope in (X, D) and $F \in \mathfrak{B}(X, Y)$, $(f|_{F(P)})(F|_P) : P \multimap P$ has a fixed point $x_0 \in P \subset X$. Since $x_0 \in (fF)x_0$ and $f^{-}x_0 \subset Gx_0$, we have $Fx_0 \cap Gx_0 \neq \emptyset$. This completes our proof.

Remark. In case $\mathfrak{B}$ is $\mathfrak{A}_c^{\kappa}$, Theorem 1 reduces to Park [3, Theorem 2], which includes earlier works of Browder, Granas and Liu, Lassonde [29], and Park *et al.* [30]. In case when $X = D$ and $F \in \mathfrak{K}(X, Y)$ is closed, Theorem 1 reduces to Chang and Yen [27, Theorem 4]. For some applications of Theorem 1, see [3] and [27].

4. A MATCHING THEOREM AND A KKM THEOREM

As in [3], Theorem 1 has a lot of equivalent formulations which constitute the foundations of the KKM theory. In this section, we give only two of them.

The following is a Ky Fan type matching theorem:

THEOREM 2. *Let* (X, D) *be a convex space,* Y *a Hausdorff space,* $T : D \multimap Y$*, and* $F \in \mathfrak{B}(X, Y)$ *a compact map. Suppose that*

(1) *for each* $x \in D$, Tx *is open; and*

(2) $\overline{F(X)} \subset T(D)$.

Then there exists an $M \in \langle D \rangle$ *such that* $F(\operatorname{co} M) \cap \bigcap\{Tx : x \in M\} \neq \emptyset$.

Proof. For each $y \in Y$, let $G^{-}y = \operatorname{co} T^{-}y$ which is the minimal D-convex set containing $T^{-}y$. This defines a map $G : X \multimap Y$ such that, for each $y \in F(X)$, $G^{-}y$ is D-convex. Moreover, for each $y \in \overline{F(X)}$, there exists an $x \in D$ such that $y \in Tx = \operatorname{Int} Tx \subset \operatorname{Int} Gx$. Therefore, all of the requirements of Theorem 1 are satisfied. Hence, there exists a coincidence point $x_0 \in X$ of F and G. For $y \in Fx_0 \cap Gx_0$, we have $x_0 \in G^{-}y = \operatorname{co} T^{-}y$, and hence there exists an $M = \{x_1, x_2, \ldots, x_n\} \in \langle T^{-}y \rangle \subset \langle D \rangle$ such that $x_0 \in \operatorname{co} M$. Since $x_i \in T^{-}y$ implies $y \in Tx_i$ for all i, $1 \leq i \leq n$, we have $y \in Fx_0 \cap \bigcap_{i=1}^n Tx_i$. This completes our proof.

Theorem 2 can be reformulated to the following KKM theorem:

THEOREM 3. *Let (X, D) be a convex space, Y a Hausdorff space, and $F \in \mathfrak{B}(X, Y)$ a compact map. Let $S : D \multimap Y$ be a map such that*

(1) *for each $x \in D$, Sx is closed in Y; and*

(2) *for any $N \in \langle D \rangle$, $F(\operatorname{co} N) \subset S(N)$.*

Then $\overline{F(X)} \cap \bigcap\{Sx : x \in D\} \neq \emptyset$.

Proof. Suppose that the conclusion does not hold. Then $\overline{F(X)} \subset T(D)$ where $Tx = Y \backslash Sx$ for $x \in D$. Therefore, by Theorem 2, there exists an $M \in \langle D \rangle$ such that $F(\operatorname{co} M) \not\subset S(M)$. This contradicts (2).

Remark. The origin of Theorem 3 is the well-known Knaster-Kuratowski-Mazurkiewicz theorem. Theorem 3 includes many extensions of the KKM theorem due to Fan, Lassonde, Park, and others. See [3]. Theorem 3 also shows that a compact map in $\mathfrak{B}$ belongs to $\mathfrak{K}$.

5. FIXED POINTS OF COMPACT MAPS

As another applications of Theorem 1, in this section, we derive some fixed point theorems for compact maps in the class $\mathfrak{B}$ and a related class $\mathfrak{B}^\sigma$.

From Theorem 1, we have the following:

THEOREM 4. *Let X and C be nonempty convex subsets of a locally convex t.v.s. E, and $F \in \mathfrak{B}(X, X + C)$ a closed compact map. Suppose that one of the following conditions holds:*

(i) *X is closed and C is compact.*

(ii) *X is compact and C is closed.*

(iii) $C = \{0\}$.

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Proof. Let V be an open convex neighborhood of the origin 0 in E, and Y a compact set satisfying $F(X) \subset Y \subset X + C$. Define $G : X \multimap Y$ by $Gx = (x + C + V) \cap Y$ for $x \in X$. Then each Gx is open in Y and $G^- y = (y - C - V) \cap X$ is convex for each $y \in Y$. Moreover, since $Y \subset X + C$, for every $y \in Y$, there exists an $x \in X$ such that $y \in x + C + V$; that is, $\{Gx : x \in X\}$ covers Y. Therefore, by Theorem 1, there exist $x_V \in X$ and $y_V \in Y$ such that $y_V \in Fx_V \cap Gx_V$; that is, $y_V - x_V \in C + V$. In other words, we obtain the assertion:

($*$) for each neighborhood V of 0 in E,

$$(F - i)(X) \cap (C + V) \neq \emptyset,$$

where $i : X \to E$ is the inclusion. Now we consider Cases (i)-(iii).

Case (i). Since X is closed, so is $(F - i)(X)$. Since C is compact and E is regular, ($*$) implies $(F - i)(X) \cap C \neq \emptyset$.

Case (ii). Since $(F - i)(X)$ is compact and C is closed, as in Case (i), we have the conclusion.

Case (iii). By $(*)$, for each neighborhood V of 0 in E, there exist $x_V,\ y_V \in X$ such that $y_V \in Fx_V$ and $y_V - x_V \in V$. Since $F(X)$ is relatively compact, we may assume that y_V converges to some $\hat{x}$. Then x_V also converges to $\hat{x}$. Since the graph of F is closed in $X \times \overline{F(X)}$, we have $\hat{x} \in F\hat{x}$.

This completes our proof.

Remark. Theorem 4 includes Lassonde [29, Theorem 1.6 and Corollary 1.18] for $\mathfrak{B} = \mathbb{K}_c$, Park *et al.* [30, Theorem 2] for $\mathfrak{B} = \mathbb{V}_c$, Park [3, Theorem 3] for $\mathfrak{B} = \mathfrak{A}_c$, and many others.

Theorem 4(iii) can be restated as follows:

THEOREM 5. *Let X be a nonempty convex subset of a locally convex t.v.s. E. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Remark. If $F \in \mathfrak{K}(X, X)$, then Theorem 5 is due to Chang and Yen [27, Theorem 2], and our proof is quite different from theirs. In [27], Theorem 5 was applied to approximable maps and a Fan type matching theorem different from Theorem 2. Note that Theorem 5 also extends Ben-El-Mechaiekh and Isac [32, Theorem 2] for approximable maps.

We define another class of multimaps as follows:

$F \in \mathfrak{B}^\sigma(X, Y) \Longleftrightarrow$ for any σ-compact D-convex subset K of (X, D), there is a closed map $\Gamma \in \mathfrak{B}(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Note that the class $\mathfrak{A}_c^\sigma$ is an example of $\mathfrak{B}^\sigma$. See [3].

THEOREM 6. *Let X be a nonempty convex subset of a locally convex t.v.s. E. Then any compact map $F \in \mathfrak{B}^\sigma(X, X)$ has a fixed point.*

Proof. Let $K = \operatorname{co} \overline{F(X)}$. Then $K \subset X$ since $\overline{F(X)} \subset X$ and X is convex. Also M is σ-compact [29, Proposition 1(3)]. Since $F \in \mathfrak{B}^\sigma(X, X)$, there exists a closed map $\Gamma \in \mathfrak{B}(K, K)$ such that $\Gamma x \subset Fx$ for all $x \in K$. Since Γ is compact and K is a nonempty convex subset of E, by Theorem 5, Γ has a fixed point $x_0 \in K$; that is, $x_0 \in \Gamma x_0 \subset Fx_0$. This completes our proof.

Remark. For $\mathfrak{B}^\sigma = \mathfrak{A}_c^\sigma$, Theorem 6 reduces to Park [3, Theorem 4], which includes numerous well-known particular cases.

6. FIXED POINTS OF CONDENSING MAPS

In this section, we deduce two new theorems on condensing maps.

Let E be a t.v.s. and C a lattice with a least element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if X is relatively compact;

(2) $\Phi(\overline{\operatorname{co}}\, X) = \Phi(X)$; and

(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

It follows that $X \subset Y$ implies $\Phi(X) \leq \Phi(Y)$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined in terms of a family of seminorms or a norm. For details, see [33, 34].

For $X \subset E$, a map $T : X \multimap E$ is said to be *Φ-condensing* provided that if $A \subset X$ and $\Phi(A) \leq \Phi(T(A))$, then A is relatively compact; that is, $\Phi(A) = 0$.

From now on, we assume that Φ is a measure of noncompactness on the given t.v.s. E if necessary.

Note that each map defined on a compact set is Φ-condensing. If E is locally convex, then a compact map $T : X \multimap E$ is γ- or χ- condensing whenever X is complete or E is quasi-complete.

The following is well-known by many authors. For example [35, 36]:

LEMMA. *Let X be a nonempty closed convex subset of a t.v.s. E and $T : X \multimap X$ a Φ-condensing map. Then there exists a nonempty compact convex subset K of X such that $T(K) \subset K$.*

From Theorem 5 and Lemma, we have the following:

THEOREM 7. *Let X be a nonempty closed convex subset of a locally convex t.v.s. E. Then any closed Φ-condensing map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Proof. By Lemma, there is a nonempty compact convex subset K of X such that $F(K) \subset K$. It is easy to check that $F|_K \in \mathfrak{B}(K, K)$ and $F|_K$ is closed. Therefore, by Theorem 5, $F|_K$ has a fixed point. This completes our proof.

We define one more class of multimaps as follows:

$F \in \mathfrak{B}^\kappa(X, Y) \iff$ for any compact D-convex subset K of (X, D), there is a closed map $\Gamma \in \mathfrak{B}(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Note that the classes $\mathfrak{A}_c^\kappa$ and $\mathfrak{B}^\sigma$ are examples of $\mathfrak{B}^\kappa$.

From Theorem 5 and Lemma, we have the following:

THEOREM 8. *Let X be a nonempty closed convex subset of a locally convex t.v.s. E. Then any Φ-condensing map $F \in \mathfrak{B}^\kappa(X, X)$ has a fixed point.*

Proof. By Lemma, there is a nonempty compact convex subset K of X such that $F(K) \subset K$. Since $F \in \mathfrak{B}^\kappa(X, X)$, there exists a closed map $\Gamma \in \mathfrak{B}(K, K)$ such that $\Gamma x \subset Fx$ for all $x \in K$. Since Γ is compact, by Theorem 5, Γ has a fixed point $x_0 \in K$; that is, $x_0 \in \Gamma x_0 \subset Fx_0$. This completes our proof.

Remark. Since Theorems 7 and 8 are concerned with very large classes of maps, they include a large number of known results. See [13].

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