COLLECTIVELY FIXED POINTS AND EQUILIBRIUM POINTS OF ABSTRACT ECONOMIES

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Abstract. From a fixed point theorem for not-necessarily locally convex topological vector spaces, we deduce a collectively fixed point theorem with applications to existence of equilibrium points and maximal elements of an abstract economy. Consequently, some known results are extended.

Key Words and Phrases. Admissible set (in the sense of Klee), (collectively) fixed point theorem, abstract economy, equilibrium point, maximal element, qualitative game.

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Usually, the existence of equilibrium points or maximal elements of abstract economies are given for locally convex topological vector spaces by using appropriate fixed point theorem. In this paper, we obtain such results for not-necessarily locally convex spaces by applying a fixed point theorem due to the author [9].

Throughout this paper, t.v.s. means Hausdorff topological vector spaces, and co, cl, and Int denote the convex hull, closure, and interior, respectively.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [8]) provided that, for every compact subset K of X and every neighborhood V of

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the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Moreover, any locally convex subset of an F-normable t.v.s. and any compact convex locally convex subset of a t.v.s. are admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Hadžić [4], Weber [12], and references therein.

The following is a particular form of our new fixed point theorem in [9]:

Theorem 1. *Let D be a convex subset of a t.v.s. E, K a compact subset of D, and $f : D \to K$ a continuous map. If $\operatorname{cl} f(D)$ is admissible in E, then f has a fixed point in K.*

From Theorem 1, we deduce the following collectively fixed point theorem:

Theorem 2. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, and K_i a nonempty compact subset of X_i such tht $K = \prod_{i\in I} K_i$ is admissible in $E = \prod_{i\in I} E_i$. For each $i \in I$, let $S_i, T_i : X = \prod_{i\in I} X_i \multimap K_i$ be maps satisfying*

(1) *for each $x \in X$, $\operatorname{co} S_i(x) \subset T_i(x)$; and*

(2) $D := \operatorname{co} K \subset \bigcup_{y\in K_i} \operatorname{Int} S_i^{-1}(y)$.

Then there exists an $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$, where $\overline{x}_i$ is the projection of $\overline{x}$ in X_i, for each $i \in I$.

Proof. Since $K = \prod_{i\in I} K_i$ is compact in E, $D = \operatorname{co} K$ is σ-compact and hence Lindelöf. Since D is regular, we know that D is paracompact. Consider $S_i|_D : D \multimap K_i$. Note that $D = \bigcup_{y\in K_i}(\operatorname{Int} S_i^{-1}(y)) \cap D$ by (2) and $\operatorname{Int}_D(S_i|_D)^{-1}(y) = (\operatorname{Int} S_i^{-1}(y)) \cap D$ for $y \in K_i$. Therefore, $(\operatorname{co} S_i)|_D : D \multimap X_i$ has a continuous selection $s_i : D \multimap K_i$ such that $s_i(x) \in \operatorname{co} S_i(x) \subset T_i(x)$ for each $x \in D$. This follows from the selection theorem of Horvath [5, Theorem 3.2] (which extends results in [1], [3], [13], [14]). Define $s : D \to K$ by $(s(x))_i = s_i(x)$ for each $i \in I$ and $x \in D$. Then s is continuous. Therefore, by Theorem 1, s has a fixed point $\overline{x} \in K$; that is, $\overline{x} \in s(\overline{x})$ and $\overline{x}_i = (s(\overline{x}))_i = s_i(\overline{x}) \in T_i(\overline{x})$. This completes our proof.

Remark. If each E_i is locally convex, then Theorem 1 reduces to Yannelis and Prabhakar [14, Theorem 3.2], Ding *et al.* [3, Theorem 2], and Husain and Tarafdar [6, Theorem 2.2], and Wu and Shen [13, Theorem 2].

We apply Theorem 2 to the existence of equilibrium points and maximal elements of an abstract economy.

An abstract economy $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ consists of an index set I of agents, a choice set X_i in a t.v.s., constraint correspondences $A_i, B_i : X = \prod_{i\in I} X_i \multimap X_i$, and a preference correspondence $P_i : X \multimap X_i$ for each $i \in I$. An equilibrium point $x = (x_i)_{i\in I} \in X$ is the one satisfying $x_i \in B_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for each $i \in I$. We say that $x \in X$ is a maximal point of the game $(X_i, P_i)_{i\in I}$ if $P_i(x) = \emptyset$ for each $i \in I$.

Theorem 3. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that, for each $i \in I$,*

(1) *X_i is a convex subset of a t.v.s. E_i and K_i is a nonempty compact subset of X_i such that $K = \prod_{i\in I} K_i$ is admissible in $E = \prod_{i\in I} E_i$;*
(2) *for each $x \in X$, $\operatorname{co} A_i(x) \subset B_i(x) \subset K_i$;*
(3) *$D := \operatorname{co} K \subset \bigcup_{y\in K_i} \operatorname{Int}(A_i^{-1}(y) \cap (P_i^{-1}(y) \cup F_i))$, where $F_i := \{x \in X : A_i(x) \cap P_i(x) = \emptyset\}$; and*
(4) *for each $x = (x_i)_{i\in I} \in X$, $x_i \notin \operatorname{co} P_i(x)$.*

Then Γ has an equilibrium point in K.

Proof. Let

$$G_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} \quad \text{for each} \quad i \in I.$$

For each $i \in I$, we define two maps $S_i, T_i : X \multimap K_i$ by

$$S_i(x) = \begin{cases} A_i(x) \cap \operatorname{co} P_i(x) & \text{if } x \in G_i, \\ A_i(x) & \text{if } x \in F_i, \end{cases}$$
$$T_i(x) = \begin{cases} B_i(x) \cap \operatorname{co} P_i(x) & \text{if } x \in G_i \\ B_i(x) & \text{if } x \in F_i. \end{cases}$$

Then for each $i \in I$ and $x \in X$, we have $\operatorname{co} S_i(x) \subset T_i(x)$; and for each $y \in K_i$, we have

$$\begin{aligned} S_i^{-1}(y) &= [(A_i^{-1}(y) \cap (\operatorname{co} P_i)^{-1}(y)) \cap G_i] \cup [A_i^{-1}(y) \cap F_i] \\ &\supset [(A_i^{-1}(y) \cap P_i^{-1}(y)) \cap G_i] \cup [A_i^{-1}(y) \cap F_i] \\ &= [A_i^{-1}(y) \cap P_i^{-1}(y)] \cup [A_i^{-1}(y) \cap F_i] \\ &= A_i^{-1}(y) \cap (P_i^{-1}(y) \cup F_i), \end{aligned}$$

which implies $D = \operatorname{co} K \subset \bigcup_{y \in K_i} \operatorname{Int} S_i^{-1}(y)$ by (3). Therefore, all of the requirements of Theorem 2 are satisfied. Hence, there exists an $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$. By (4), $\overline{x}_i \notin \operatorname{co} P_i(\overline{x})$. Therefore, $\overline{x}_i \in B_i(\overline{x})$ for each $i \in I$ by the definition of T_i and hence $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$. This shows that $\overline{x}$ is an equilibrium point of Γ.

Remark. We followed the proof of Tarafdar [10, Theorem 3.1]. If all of E_i's are locally convex, then Theorem 3 reduces to Ding *et al.* [3, Theorems 4 and 5] and Husain and Tarafdar [6, Theorem 3.1]. Moreover, condition (3) seems to be artificial and is implied by

(3)′ $\quad D := \operatorname{co} K \subset \bigcup_{y \in K_i} \operatorname{Int}(A_i^{-1}(y) \cap P_i^{-1}(y))$.

In this case, for locally convex E_i's, Theorem 3 sharpens Wu and Shen [13, Theorem 10], which in turn extends results of Yannelis and Prabhakar [14], S.-Y. Chang [2], Tian [11], and Im *et al.* [7]. Similarly, [13, Theorems 11 and 12] can also be improved.

Theorem 4. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that, for each $i \in I$,*

(1) *X_i is a convex subset of a t.v.s. E_i and K_i is a nonempty compact convex subset of X_i such that $K = \prod_{i \in I} K_i$ is admissible in $E = \prod_{i \in I} E_i$;*
(2) *$K \subset \bigcup_{y \in K_i} \operatorname{Int}_X (P_i^{-1}(y) \cup F_i)$, where $X = \prod_{i \in I} X_i$ and $F_i = \{x \in X : P_i(x) = \emptyset\}$; and*
(3) *for each $x = (x_i)_{i \in I} \in X$, $x_i \notin \operatorname{co} P_i(x)$.*

Then Γ has a maximal element in K.

Proof. For each $i \in I$, define a map $A_i : X \multimap K_i$ by $A_i(x) = K_i$ for $x \in X$. Now we can apply Theorem 3 with $A_i \equiv B_i$ and the conclusion follows.

Remark. If each E_i is locally convex, then Theorem 4 reduces to Husain and Tarafdar [6, Theorem 3.2].

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