ANOTHER GENERALIZATIONS OF THE EKELAND TYPE VARIATIONAL PRINCIPLES

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Abstract. We extend and unify some equivalent formulations of the Ekeland variational principle due to Oettli and Théra [OT], Blum and Oettli [BO], Kada *et al.* [KS], and Park and Kang [PK].

Key Words and Phrases. W-distance, quasi-order $\preccurlyeq$, quasi-metric, $\preccurlyeq$-complete, lower semicontinuous, the Caristi-Kirk-Browder fixed point theorem.

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1. Introduction

In a recent work of the author and B. G. Kang [PK], known extensions or equivalent formulations of Ekeland's variational principle were unified in a far-reaching general theorem. Since then, there also have appeared some other extensions of the principle.

In fact, Oettli and Théra [OT] obtained another extended equivalent form of the principle as well as some applications. In the same spirit, Blum and Oettli [BO] gave an extension of Takahashi's nonconvex minimization theorem [T]. Moreover, Kada, Suzuki, and Takahashi [KS] improved Takahashi's theorem replacing the involved metric by a newly defined W-distance and gave some applications.

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The aim of this paper is to unify the results in [OT], [BO], [KS] along the lines of [PK] and to improve the equivalent formulations of Ekeland's principle in various aspects.

2. Main results

Kada *et al.* [KS] introduced the concept of W-distances for a metric space (X, d) as follows:

A function $\omega : X \times X \to [0, \infty)$ is called a W-*distance* on X if the following are satisfied:

(1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$ for any $x, y, z \in X$;
(2) for any $x \in X$, $\omega(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous; and
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

In [KS], many examples and properties of W-distances were given.

Let X be a nonempty set and $\preccurlyeq$ a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on X. Let $S(x) = \{y \in X \mid x \preccurlyeq y\}$ for $x \in X$, and $\leq$ be the usual order in the extended real number system $[-\infty, \infty]$.

Let d be a quasi-metric (that is, not necessarily symmetric) on X. Then for the quasi-metric space (X, d), the concepts of W-distances, Cauchy sequences, completeness, and Banach contractions can be defined.

In a quasi-metric space (X, d) with a quasi-order $\preccurlyeq$, a set $S(a)$ for some $a \in X$ is said to be $\preccurlyeq$-*complete* if every nondecreasing Cauchy sequence in $S(a)$ converges. For details, see [PK] and references therein.

In this paper, let $\phi : X \times X \to (-\infty, \infty]$ be a function such that

(4) $\phi(x, \cdot)$ is lower semicontinuous for all $x \in X$;
(5) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ for all $x, y, z \in X$; and
(6) there exists an $x_0 \in X$ such that $\inf_{y \in X} \phi(x_0, y) > -\infty$.

The following is our main result in this paper:

Theorem. *Let (X,d) be a quasi-metric space. Let $\omega : X\times X \to [0,\infty)$ be a W-distance on X and $\phi : X\times X \to (-\infty,\infty]$ a function satisfying* (4)-(6)*. Define a quasi-order $\preccurlyeq$ on X by*

$$x \preccurlyeq y \quad \textit{iff} \quad x = y \quad \textit{or} \quad \phi(x,y)+\omega(x,y) \le 0.$$

Suppose that there exists a $u\in X$ such that $\inf_{y\in X}\phi(u,y) > -\infty$ and $S(u)=\{y\in X \mid u\preccurlyeq y\}$ is $\preccurlyeq$-complete.

Then the following statements hold.

(i) *There exists a maximal point $v\in S(u)$; that is,*

$$\forall\, w\in X\backslash\{v\},\ \phi(v,w)+\omega(v,w) > 0.$$

(ii) *If $T : S(u)\to 2^X$ satisfies the condition*

$$\forall\, x\in S(u)\backslash T(x) \quad \exists\, y\in X\backslash\{x\} \quad \textit{such that } x\preccurlyeq y,$$

then T has a fixed point $v\in S(u)$; that is, $v\in T(v)$.

(iii) *A function $f : S(u)\to X$ satisfying $x\preccurlyeq f(x)$ for all $x\in S(u)$ has a fixed point.*

(iv) *If $T : S(u)\to 2^X\backslash\{\emptyset\}$ satisfies the condition*

$$\forall x\in S(u),\ \forall\, y\in T(x),\ x\preccurlyeq y \textit{ holds},$$

then T has a stationary point $v\in S(u)$; that is, $T(v)=\{v\}$.

(v) *A family $\mathcal{F}$ of functions $f : S(u)\to X$ satisfying $x\preccurlyeq f(x)$ for all $x\in S(u)$ has a common fixed point $v\in S(u)$.*

(vi) *If Y is a subset of X such that for each $x\in S(u)\backslash Y$ there exists a $z\in S(x)\backslash\{x\}$, then there exists a $v\in S(u)\cap Y$.*

(vii) *If, for each $v\in S(u)$ with $\inf_{y\in X}\phi(v,y)<0$, there exists a $w\in S(v)\backslash\{v\}$, then there exists an $x_0\in S(u)$ such that $\inf_{y\in X}\phi(x_0,y)\ge 0$.*

In fact, (i)–(vi) *are equivalent,* (i) $\Longrightarrow$ (vii), *and* (vii) $\Longrightarrow$ (i) *whenever either* (a) $\omega(x,y)=0$ *implies* $x=y$*; or* (b) $\phi(x,x)=0$ *for all* $x\in X$.

Proof. (i) By (1) and (5), $\preccurlyeq$ is a quasi-order. We construct inductively a sequence of points $v_n\in S(u)$. To each v_n we let

$$S_n := \{v\in S(u)\,|\, v=v_n \text{ or } \phi(v_n,v)+\omega(v_n,v)\le 0\} = S(v_n)$$

and define the number

$$\gamma_n := \inf_{v\in S_n}\phi(v_n,v).$$

Note that each S_n is a closed subset of the $\preccurlyeq$-complete subset $S(u)$ by the lower semicontinuity of ϕ and ω.

Note that $v_n\in S_n\neq\emptyset$ and that $\gamma_n\le 0$. Let $u=v_0$. Then $S(u)=S_0$ and, by the hypothesis, $\gamma_0\ge\inf_{v\in X}\phi(v_0,v)>-\infty$. Let $n\ge 1$ and assume that v_{n-1} with $\gamma_{n-1}>-\infty$ is already known. Then choose $v_n\in S_{n-1}$ such that

$$\phi(v_{n-1},v_n)\le\gamma_{n-1}+\frac{1}{n}.$$

Since $v_n\in S_{n-1}$, for any $v\in S_n\backslash\{v_n\}$, we have

$$\begin{aligned}&\phi(v_{n-1},v)+\omega(v_{n-1},v)\\ \le\ &\phi(v_{n-1},v_n)+\omega(v_{n-1},v_n)+\phi(v_n,v)+\omega(v_n,v)\\ \le\ &\phi(v_n,v)+\omega(v_n,v)\le 0\end{aligned}$$

and hence $v\in S_{n-1}$; that is $S_{n-1}\supset S_n$. Therefore, we obtain

$$\begin{aligned}\gamma_n &= \inf_{v\in S_n}\phi(v_n,v)\ge\inf_{v\in S_n}(\phi(v_{n-1},v)-\phi(v_{n-1},v_n))\\ &\ge\inf_{v\in S_{n-1}}(\phi(v_{n-1},v)-\phi(v_{n-1},v_n))\\ &=\gamma_{n-1}-\phi(v_{n-1},v_n)\ge-\frac{1}{n}.\end{aligned}$$

If $v\in S_n\backslash\{v_n\}$, then

$$\omega(v_n,v)\le-\phi(v_n,v)\le-\gamma_n\le\frac{1}{n}.$$

Since $\omega(v_n, v)$ is a W-distance, for any $\varepsilon > 0$ we can choose a sufficiently large n such that

$$\omega(v_n, v) \leq \frac{1}{n} \text{ and } \omega(v_n, v') \leq \frac{1}{n} \text{ imply } d(v, v') < \varepsilon$$

for all $v, v' \in S_n$. Therefore the diameters of the sets S_n tends to zero. Moreover for all $k \geq n$ we have $v_k \in S_k \subset S_n$ and hence $d(v_n, v_k) \leq 1/n$. Thus the sequence $\{v_n\}$ is Cauchy in the $\preccurlyeq$-complete set $S(u)$ and hence converges to some $v^* \in S(u)$. Clearly we have $v^* \in \bigcap_{n=0}^{\infty} S_n$. Since the diameters of the sets S_n tends zero, we have $\bigcap_{n=0}^{\infty} S_n = \{v^*\}$. We claim that v^* is maximal. Otherwise, there exists a $w \in X \backslash \{v^*\}$ such that

$$\phi(v^*, w) + \omega(v^*, w) \leq 0.$$

Since $v^* \in S_n$ for all n and hence

$$\phi(v_n, v^*) + \omega(v_n, v^*) \leq 0$$

and

$$\phi(v_n, w) + \omega(v_n, w) \leq 0$$

by the triangle inequalities of ϕ and ω. Therefore, $w \in S_n$ for all n. Therefore, we should have $w = v^*$. This contradiction completes our proof of (i).

The equivalency of (i)-(v) can be seen as in Park [P1-5].

We now show that (i) $\iff$ (vi).

(i) $\implies$ (vi) By (i), there exists a $v \in S(u)$ such that $\phi(v, w) + \omega(v, w) > 0$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore $v \in S(u) \cap Y$.

(vi) $\implies$ (i) For all $x \in X$, let

$$A(x) := \{y \in X \,|\, x \neq y,\ \phi(x, y) + \omega(x, y) \leq 0\} = S(x) \backslash \{x\}.$$

Choose $Y = \{x \in X \,|\, A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the bypothesis of (vi) is satisfied. Therefore, by (vi), there exists a $v \in S(u) \cap Y$. Hence $A(v) = \emptyset$; that is, $\phi(v, w) + \omega(v, w) > 0$ for all $w \neq v$. Hence (i) holds.

Finally, the proofs of (i) $\implies$ (vii) and (vii) $\implies$ (i) can be given as in [PK]. This completes our proof.

Remarks. 1. The primitive versions of Theorem for $\omega = d$ and $\phi(x, y) = f(y) - f(x)$, where $f : X \to (-\infty, \infty]$ is a proper lower semicontinuous function on X bounded from below, were obtained by the following authors:

(i) Phelps [Ph] and Ekeland [E1-3].
(ii) Tuy [Tu], Penot [Pe], and Mizoguchi and Takahashi [MT].
(iii) Caristi, Kirk, and Browder [C].
(iv) Maschler and Peleg [MP].
(v) Kasahara [K].
(vii) Takahashi [T].

2. Recently, in case $\phi(x,x) = 0$ for all $x \in X$, Oettli and Théra [OT] obtained Theorem (vi) for $\omega = d$, Blum and Oettli [BO] obtained (vii) for $\omega = d$, and Kada *et al.* [KS] obtained (vii) for $\phi(x,y) = f(y) - f(x)$ as above.

3. Note that all of the above authors obtained particular forms of Theorem (i)-(vii) for complete metric spaces. Quasi-metric versions were due to Hicks [H] and Park and Kang [PK].

4. Applications of Theorem can be seen in the papers quoted above and references therein. Especially, for new applications for (vi) and (vii), see [OT], [KS].

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