FIXED POINTS OF THE BETTER ADMISSIBLE MULTIMAPS

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Abstract. We give general Schauder type fixed point theorems for compact multimaps in the "better" admissible class $\mathfrak{B}$ defined on admissible convex subsets (in the sense of Klee) of a topological vector space not necessarily locally convex. Our new theorems subsume a large number of particular forms, and generalize them in terms of the involving spaces and the multimaps as well. We apply our new results to condensing maps.

Key Words and Phrases. The Schauder fixed point theorem, multimap (map), closed map, compact map, acyclic, admissible (in the sense of Klee), topological vector space (t.v.s.).

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1. Introduction

The well-known Schauder fixed point theorem states that a continuous compact map $f : H \to H$ defined on a closed convex subset H of a Banach space has a fixed point. This theorem has an enormous influence on fixed point theory, differential equations, variational inequalities, equilibrium problems, and many other fields in mathematics. Moreover, there have appeared a large number of generalizations and their applications for various classes of multimaps defined on convex subsets of topological vector spaces more general than Banach spaces.

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Therefore, it would be desirable to present the most general result which can subsume many of generalizations of the Schauder theorem.

In this paper, we give general Schauder type fixed point theorems for compact multimaps related to the class $\mathfrak{B}$ of "better" admissible maps (see Park [P3]) defined on admissible convex subsets (in the sense of Klee [K]) of a topological vector space which is not necessarily locally convex. Our new theorems subsume a large number of particular forms, and generalize them in terms of the involving spaces and the multimaps as well. We apply our new results to condensing maps.

Further details on our study will be appearing in our forthcoming works.

2. Fixed points of compact maps

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^{-}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^{-}(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map T with closed values is closed.

Throughout this paper, t.v.s. means Hausdorff topological vector spaces, and co denotes the convex hull.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [K]) provided that, for every compact subset A of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : A \to X$ such that $x - h(x) \in V$ for all $x \in A$ and $h(A)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0, 1)$ of equivalence classes of measurable functions on $[0, 1]$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others.

Moreover, any locally convex subset of an F-normable t.v.s. and any compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Hadžić [H], Weber [W], and references therein.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

We define the *"better" admissible class* $\mathfrak{B}$ of multimaps defined on X as follows:

$F \in \mathfrak{B}(X,Y) \Longleftrightarrow F : X \multimap Y$ is a map such that for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite composites of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of t.v.s.), admissible maps of Górniewicz, permissible maps of Dzedzej, the class $\mathbb{K}_c^+$ of Lassonde, the class $\mathbb{V}_c^+$ of Park *et al.*, approximable maps of Ben-El-Mechaiekh and Idizk, and many others. Those subclasses are all examples of the admissible class $\mathfrak{A}_c^\kappa$; see Park [P1-4], Park and Kim [PK1-3]. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were given recently by Park [P5].

We define another class of multimaps containing $\mathfrak{B}(X,Y)$ as follows:

$F \in \mathfrak{B}^\sigma(X,Y) \Longleftrightarrow F : X \multimap Y$ is a map such that for any σ-compact convex subset K of X, there is a closed map $\Gamma \in \mathfrak{B}(K,Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Note that the classes $\mathbb{K}_c^+$, $\mathbb{V}_c^+$ and $\mathfrak{A}_c^\sigma$ are examples of $\mathfrak{B}^\sigma$; see Park [P1-3], Park and Kim [PK1-3].

The following is our new fixed point theorem:

Theorem 1. *Let E be a t.v.s. and X an admissible convex subset of E. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Proof. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E. Since F is closed and compact, it is sufficient to show that for any $V \in \mathcal{V}$, there exists an $x_V \in X$ such that $(x_V + V) \cap F(x_V) \neq \emptyset$.

Since $\overline{F(X)}$ is a compact subset of the admissible subset X, there exist a continuous map $h : \overline{F(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset L \cap X$. Let $M := h(\overline{F(X)})$. Then M is a compact subset of L and hence $P := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{F(X)} \to P$ and $F|_P : P \to \overline{F(X)}$. Since $F \in \mathfrak{B}(X, X)$, the composite $h(F|_P)$ has a fixed point $x_V \in P$; that is, $x_V \in hF(x_V)$ and hence $x_V = h(y)$ for some $y \in F(x_V)$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap F(x_V) \neq \emptyset$. This completes our proof.

From Theorem 1, we have the following:

Theorem 2. *Let X be an admissible convex subset of a t.v.s. E. Then any compact map $F \in \mathfrak{B}^\sigma(X, X)$ has a fixed point.*

Proof. Let $K = \operatorname{co} \overline{F(X)}$. Then $K \subset X$ since $\overline{F(X)} \subset X$ and X is convex. It is well-known that K is σ-compact. Since $F \in \mathfrak{B}^\sigma(X, X)$, there exists a closed map $\Gamma \in \mathfrak{B}(K, K)$ such that $\Gamma x \subset Fx$ for all $x \in K$. Since Γ is compact and K is an admissible convex subset of E, by Theorem 1, Γ has a fixed point $x_0 \in K$; that is, $x_0 \in \Gamma x_0 \subset F x_0$. This completes our proof.

It should be noted that, in Theorems 1 and 2, the admissibility of X can be replaced by that of $\overline{F(X)}$.

3. Fixed points of condensing maps

In this section, we deduce two new theorems on condensing maps.

Let E be a t.v.s. and C a lattice with a least element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if X is relatively compact;
(2) $\Phi(\overline{\text{co}}\, X) = \Phi(X)$; and
(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

It follows that $X \subset Y$ implies $\Phi(X) \leq \Phi(Y)$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined in terms of a family of seminorms or a norm.

For $X \subset E$, a map $T : X \multimap E$ is said to be Φ-*condensing* provided that if $A \subset X$ and $\Phi(A) \leq \Phi(T(A))$, then A is relatively compact; that is, $\Phi(A) = 0$.

From now on, we assume that Φ is a measure of noncompactness on the given t.v.s. E if necessary.

Note that each map defined on a compact set is Φ-condensing. If E is locally convex, then a compact map $T : X \multimap E$ is γ- or χ-condensing whenever X is complete or E is quasi-complete.

The following is well-known :

Lemma. *Let X be a nonempty closed convex subset of a t.v.s. E and $T : X \multimap X$ a Φ-condensing map. Then there exists a nonempty compact convex subset K of X such that $T(K) \subset K$.*

From Theorem 1 and Lemma, we have the following:

Theorem 3. *Let X be an admissible closed convex subset of a t.v.s. E. Then any closed Φ-condensing map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Proof. By Lemma, there is a nonempty compact convex subset K of X such that $F(K) \subset K$. It is easy to check that $F|_K \in \mathfrak{B}(K, K)$ and $F|_K$ is closed. Therefore, by Theorem 1, $F|_K$ has a fixed point. This completes our proof.

We define one more class of multimaps as follows:

$F \in \mathfrak{B}^\kappa(X,Y) \iff$ for any compact convex subset K of X, there is a closed map $\Gamma \in \mathfrak{B}(K,Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Note that the classes $\mathfrak{A}_c^\kappa$ and $\mathfrak{B}^\sigma$ are examples of $\mathfrak{B}^\kappa$.

From Theorem 1 and Lemma, we have the following:

Theorem 4. *Let X be an admissible closed convex subset of a t.v.s. E. Then any Φ-condensing map $F \in \mathfrak{B}^\kappa(X,X)$ has a fixed point.*

Proof. By Lemma, there is a nonempty compact convex subset K of X such that $F(K) \subset K$. Since $F \in \mathfrak{B}^\kappa(X,X)$, there exists a closed map $\Gamma \in \mathfrak{B}(K,K)$ such that $\Gamma x \subset Fx$ for all $x \in K$. Since Γ is compact, by Theorem 1, Γ has a fixed point $x_0 \in K$; that is, $x_0 \in \Gamma x_0 \subset Fx_0$. This completes our proof.

Since Theorems 3 and 4 are concerned with very large classes of maps, they include a large number of known results. See [P4].

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