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Generalized Equilibrium Problems and Generalized Complementarity Problems[1]

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Abstract. From a general minimax inequality or an abstract lopsided saddle-point theorem, we deduce general Karamardian-type equilibrium theorems and generalized complementarity theorems. Our new results extend a number of well-known earlier works of many authors.

Key Words. Multimaps, convex spaces, polytopes, upper-semicontinuous maps, acyclic spaces, admissible maps, quasiconcave functions, equilibrium problems, generalized complementarity problems, cones, polars.

1. Introduction

Recently, the author (Ref. 1, Theorem 11) obtained a general Fan-type minimax inequality related to multimaps in very general classes within the frame of the KKM theory. This was used to obtain a lopsided saddle-point theorem (Ref. 2, Theorem 3), from which we deduced fixed-point theorems for those general classes of multimaps in topological vector spaces; see Ref. 2.

In the present paper, we apply the lopsided saddle-point theorem to obtain the Karamardian-type equilibrium theorems and generalized complementarity theorems for general topological vector spaces not necessarily locally convex. Our new results extend a number of well-known earlier works and our proofs are much simpler than known ones.

A multimap or map $T: X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y. As usual, T also denotes its graph; that is,

$$(x, y) \in T, \qquad \text{iff } y \in Tx.$$

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Given two maps $T: X \multimap Y$ and $S: Y \multimap Z$, the composition $S \circ T: X \multimap Z$ is defined by

$$(S \circ T)(x) = S(T(x)), \qquad x \in X.$$

A convex space is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls will be called polytopes; see Ref. 3.

For topological spaces X and Y, a map $T: X \multimap Y$ is upper semicontinuous (u.s.c.) iff, for each closed set $B \subset Y$,

$$T^{-}(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$$

is closed in X. We recall that a nonempty topological space is acyclic iff all of its reduced Čech homology groups over rationals vanish.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $T: X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ denotes the set of finite compositions of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $T \in \mathfrak{A}_c$ is u.s.c. and compact-valued;
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in a topological vector space (in short, t.v.s.), and others; for the literature, see Refs. 1, 2, 4, 5. We define

$T \in \mathfrak{A}_c^\kappa(X, Y) \Leftrightarrow$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma(x) \subset T(x)$ for each $x \in K$.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\kappa$. Any class $\mathfrak{A}_c^\kappa$ will be called admissible; for details and examples, see Refs. 1, 2, 5. Approximable maps recently due to Ben-El-Mechaiekh and Idzik (Ref. 6) belong to $\mathfrak{A}_c^\kappa$. A compact u.s.c. map defined on a closed subset of a locally convex Hausdorff t.v.s. with closed values is approximable whenever its values are all convex, contractible, decomposable, or ∞-proximally connected; see Ref. 6.

Recall that a real-valued function $f: X \to \mathbb{R}$ defined on a convex space X is said to be quasiconcave [resp., quasiconvex] whenever $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is convex for each $r \in \mathbb{R}$.

2. Generalized Equilibrium Problems

We begin with the following lopsided saddle-point theorem; see Ref. 2, Theorem 3.

Theorem 2.1. Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathfrak{A}_c^\kappa(X, Y)$. Let $\alpha : X \times Y \to \mathbb{R}$ be a continuous real function such that, for each $y \in Y$, $x \mapsto \alpha(x, y)$ is quasiconvex on X. Then, there exists an $(x_0, y_0) \in T$ such that

$$\alpha(x_0, y_0) \leq \alpha(x, y_0), \qquad \forall x \in X.$$

Note that Theorem 2.1 was applied to obtain numerous results on best approximations and fixed points; for the literature, see Ref. 2. Note also that, if T is single-valued, the Hausdorff assumption on Y is not necessary; see Ref. 1.

From Theorem 2.1, we deduce the following generalized equilibrium theorem.

Theorem 2.2. Let X be a Hausdorff compact convex space, Z a Hausdorff space, and $T \in \mathfrak{A}_c^\kappa(X, Z)$. Let $\phi : X \times X \times Z \to \mathbb{R}$ be a continuous function such that, for each $(y, z) \in X \times Z$, $x \mapsto \phi(x, y, z)$ is quasiconvex on X. Then, there exist an $x_0 \in X$ and a $z_0 \in T(x_0)$ such that

$$\phi(x_0, x_0, z_0) \leq \phi(x, x_0, z_0), \qquad \forall x \in X.$$

Proof. Put $Y = X \times Z$ and define $T' : X \multimap Y$ by

$$T'(x) = \{x\} \times T(x), \qquad \forall x \in X.$$

Then, $T' \in \mathfrak{A}_c^\kappa(X, Y)$. Now, we use Theorem 2.1 with T' replacing T and

$$\phi(x, y, z) = \alpha(x, (y, z)), \qquad x \in X, \quad (y, z) \in Y = X \times Z.$$

Then, there exist an $x_0 \in X$ and a $(y_0, z_0) \in T'(x_0)$ satisfying

$$\phi(x_0, y_0, z_0) \leq \phi(x, y_0, z_0), \qquad \forall x \in X.$$

However,

$$(y_0, z_0) \in T'(x_0) = \{x_0\} \times T(x_0) \text{ implies } x_0 = y_0 \text{ and } z_0 \in T(x_0).$$

This completes our proof. □

Example 2.1. See Saigal (Ref. 7, Lemma 4.1). Here, X is a subset of $\mathbb{R}^n = Z$, $T \in \mathbb{V}(X, \mathbb{R}^n)$ with contractible values, and $\phi(x, y, z) = \langle x - y, z \rangle$.

Example 2.2. See Parida and Sen (Ref. 8, Theorem 1). Here, X is a subset of $\mathbb{R}^n$, Z a closed convex set in $\mathbb{R}^p$, and $T \in \mathbb{K}(X, Z)$.

From Theorem 2.1, we obtain the following Karamardian-type equilibrium theorem.

Theorem 2.3. Let X be a compact convex space, let Z be any set, and let $\psi: X \times Z \to \mathbb{R}$ and $f: X \to Z$ be functions such that

(i) $\forall z \in Z$, $\psi(\cdot, z)$ is quasiconvex on X;
(ii) $(x, y) \mapsto \psi(x, f(y))$ is continuous on $X \times X$.

Then, there exists an $\hat{x} \in X$ such that

$$\psi(\hat{x}, f(\hat{x})) \leq \psi(x, f(\hat{x})), \qquad \forall x \in X.$$

Proof. Let $X = Y$ and $T = 1_X$, the identity map on X, in Theorem 2.1. Let $\alpha: X \times X \to \mathbb{R}$ be defined by

$$\alpha(x, y) = \psi(x, f(y)).$$

Then, $\forall y \in X$, $\alpha(\cdot, y) = \psi(\cdot, f(y))$ is quasiconvex on X by (i); and $(x, y) \mapsto \alpha(x, y)$ is continuous on $X \times X$ by (ii). Therefore, by Theorem 2.1, there exists an $(x_0, y_0) \in T$ with $x_0 = y_0 \in X$ such that

$$\alpha(x_0, x_0) \leq \alpha(x, x_0), \qquad \forall x \in X.$$

This completes our proof. □

Example 2.3. See Hartman and Stampacchia (Ref. 9, Lemma 1). Here, $X \subset \mathbb{R}^n = Z$, $f: X \to \mathbb{R}^n$ is continuous, and $\psi(x, f(y)) = \langle f(y), x - y \rangle$.

Example 2.4. See Browder (Ref. 10, Theorem 3 and Ref. 11, Theorem 2). Here, X is a subset of a t.v.s. E, $Z = E^*$, $f: X \to E^*$ is continuous, and $\psi(x, f(y)) = \langle f(y), x - y \rangle$.

Example 2.5. See Lions and Stampacchia (Ref. 12), Stampacchia (Ref. 13), and Mosco (Ref. 14). Here, X is a subset of an inner product space V, $a: V \times V \to \mathbb{R}$ is a continuous bilinear form on V, $Z = V^*$, $f: X \to V^*$ is a constant map with the value $p \in V^*$, and

$$\psi(x, f(y)) = \langle p, x - y \rangle - a(x, x - y).$$

Example 2.6. See Fan (Ref. 15, Theorem 2). Here, X is a subset of a normed vector space E, $Z=E$, $f: X\to E$ is a continuous map, and $\psi(x, f(y)) = \|x-f(y)\|$.

Example 2.7. See Karamardian (Ref. 17, Lemma 3.2). Here, X is a subset of a locally convex t.v.s. and Z is a vector space.

Example 2.8. See Juberg and Karamardian (Ref. 16, Lemma). Here, X is a subset of a locally convex t.v.s. E, Z a t.v.s., and $\psi(x, f(y)) = \langle x-y, f(y)\rangle$, where $\langle\, ,\, \rangle$ is a real bilinear functional on $E\times Z$.

Example 2.9. See Yao (Ref. 18, Lemma 3.1). Here, X is a subset of $\mathbb{R}^n$, $f: \mathbb{R}^n\to\mathbb{R}^n$ is continuous, $g: \mathbb{R}^n\to\mathbb{R}^n$ is linear, and $\psi(x, f(y)) = \langle g(x)-g(y), f(y)\rangle$.

Example 2.10. See Parida, Sahoo, and Kumar (Ref. 19, Theorem 3.1), Behera and Panda (Ref. 20, Theorem 2.2), Siddiqi, Khaliq, and Ansari (Ref. 21, Theorem 3.2). Here, X is a subset of a t.v.s. E, $Z=E^*$, $f: X\to E^*$ and $\eta: X\times X\to E$ are continuous maps such that $\eta(\cdot, y)$ is linear for $y\in X$, and $\psi(x, f(y)) = \langle f(y), \eta(x, y)\rangle$, where $\langle\, ,\, \rangle$ is the pairing on $E^*\times E$.

The following is the noncompact version of Theorem 2.3:

Theorem 2.4. Let X be a convex space, K a nonempty compact subset of X, Z a set, let $\psi: X\times Z\to\mathbb{R}$ and $f: X\to Z$ be functions. Suppose that:

(i) $\forall z\in Z$, $\psi(\cdot, z)$ is quasiconvex on X;
(ii) $(x, y)\mapsto\psi(x, f(y))$ is continuous on $X\times X$;
(iii) for each nonempty finite subset N of X, there exists a compact convex subset L_N of X containing N such that, for each $x\in L_N\setminus K$,

$$\exists u\in L_N,\ \psi(x, f(x)) > \psi(u, f(x)).$$

Then, there exists an $\hat{x}\in K$ such that

$$\psi(\hat{x}, f(\hat{x})) \le \psi(x, f(\hat{x})), \qquad \forall x\in X.$$

Moreover, the set of solutions $\hat{x}$ is a compact subset of K.

Proof. For every $u\in X$, let

$$K_u = \{x\in K: \psi(x, f(x)) \le \psi(u, f(x))\},$$

which is a closed subset of K by (ii). We claim that

$$\bigcap_{u\in N} K_u \neq \emptyset,$$

for each nonempty finite subset N of X. By (iii), there exists a compact convex subset L_N of X containing N. Then by Theorem 2.3, there exists an $\hat{x}\in L_N$ such that

$$\psi(\hat{x}, f(\hat{x})) \leq \psi(u, f(\hat{x})), \qquad \forall u\in L_N.$$

Since $N\subset L_N$, this inequality holds for all $u\in N$. Note that $\hat{x}\in K$ by (iii), and hence,

$$\hat{x}\in \bigcap_{u\in N} K_u.$$

Since K is compact, we have

$$\bigcap_{u\in X} K_u \neq \emptyset.$$

Note that the set of the required solutions $\hat{x}$ is the intersection of closed subsets in the compact set K. This completes our proof. □

Example 2.11. See Karamardian (Ref. 17, Theorem 3.1), Juberg and Karamardian (Ref. 16, Theorem). Here, X is a subset of a locally convex t.v.s. E, Z is a t.v.s., and $\psi(x, f(y)) = \langle x, f(y)\rangle$, where $\langle\ ,\ \rangle$ is a real bilinear functional on $E\times Z$, under the restriction that:

(iii)′ there exists a nonempty compact convex subset L in X with the property that, for each $x\in X\backslash L$, there exists a $u\in L$ such that $\langle x-u, f(x)\rangle > 0$,

instead of the existence of the compact set K satisfying (iii).

Note that (iii)′ $\Rightarrow$ (iii), and not conversely.

Example 2.12. See Siddiqi, Khaliq, and Ansari (Ref. 21, Theorem 3.3). Here, X is a closed convex subset of a t.v.s. E, $Z=E^*$, $f: X\to E^*$ and $\eta: X\times X\to E$ are continuous maps such that $\eta(\cdot, y)$ is linear for $y\in X$, and $\psi(x, f(y)) = \langle f(y), \eta(x, y)\rangle$, where $\langle\ ,\ \rangle$ is the pairing of $E^*\times E$, under the restriction corresponding to (iii)′.

Remark 2.1. The result of Karamardian and Juberg in Example 2.11 was extended by the author (Ref. 22, Theorem 1) in another direction, which was applied to obtain many theorems, including that of Brouwer; see Ref. 22.

3. Generalized Complementarity Problem

Let E be a t.v.s., let F be a vector space, and let $\langle\ ,\ \rangle: E\times F\to\mathbb{R}$ be a bilinear form. A nonempty subset X of E is called a cone iff

$$\alpha x+\beta y\in X, \qquad \forall\alpha, \beta\geq 0 \text{ and } x, y\in X.$$

The polar X^* of a cone X is the cone in F defined by

$$X^*=\{p\in F: \langle x, p\rangle\geq 0, \forall x\in X\}.$$

Then, the generalized complementarity problem (GCP) for a function $f: X\to Y$ is to find an $x_0\in X$ satisfying

$$f(x_0)\in X^*, \qquad \langle x_0, f(x_0)\rangle=0. \tag{1}$$

From Theorem 2.4, we have the following theorem.

Theorem 3.1. The GCP, as given by (1), has a solution whenever:

(i) the function $(x, y)\mapsto\langle x, f(y)\rangle$ is continuous on $X\times X$;
(ii) there is a nonempty compact subset K of X such that, for each nonempty finite subset N of X, we have a compact convex subset L_N of X containing N such that, for each $x\in L_N\backslash K$,

$$\langle x-u, f(x)\rangle>0, \qquad \text{for some } u\in L_N.$$

Proof. Put $Z=F$ and $\psi(x, f(y))=\langle x, f(y)\rangle$ in Theorem 2.4. Then, all of the requirements are satisfied. Therefore, there exists an $x_0\in K$ such that

$$\langle x, f(x_0)\rangle\geq\langle x_0, f(x_0)\rangle, \qquad \forall x\in X.$$

This implies that

$$\langle \alpha x, f(x_0)\rangle\geq\langle x_0, f(x_0)\rangle, \qquad \forall\alpha>1 \text{ and } \forall x\in X.$$

By considering $x=x_0$, we have

$$\langle x_0, f(x_0)\rangle\geq 0.$$

On the other hand, by considering $x=0$, we have

$$\langle x_0, f(x_0)\rangle\leq 0.$$

Therefore,

$$\langle x_0, f(x_0)\rangle=0$$

and

$$\langle x, f(x_0)\rangle \geq 0, \qquad \forall x \in X,$$

whence we have $f(x_0) \in X^*$. This completes our proof. □

Example 3.1. See Karamardian (Ref. 17, Theorem 3.1). Here, X is a cone of a locally convex t.v.s. E, Z is a t.v.s., and the restriction (iii)′ applies instead of (ii).

Example 3.2. See Park (Ref. 22, Corollary 2.2). Here, $f: X \to E^*$ is continuous. This result was obtained by different argument.

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