ON THE BROWDER TYPE BEST APPROXIMATION THEOREMS

Ming-Po Chen* **
Institute of Mathematics
Academia Sinica
Nankang, Taipei 11529, Taiwan
E-MAIL: MAAPO@ccvax.sinica.edu.tw

Sehie Park
Department of Mathematics Seoul National University
Seoul 151-742, Korea

Abstract. Best approximation theorems extending Browder's sharpened form of the Schauder fixed point theore
are obtained. Our new results extend and unify many known theorems.

1. INTRODUCTION

Recently, Sehgal and Singh [6] obtained a theorem on the best approximation of a continuous f
with respect to a generalized affine type map. This result provides extensions of some well-known fixe
theorems due to Browder [1] and others. Note that their method is based on the KKM principle.

On the other hand, there have appeared some best approximation or fixed point theorems for maps
domains and ranges have different topologies. For the literature, see [2]. Motivated by those new
the second author [5] obtained general theorems unifying known Fan or Prolla type best applied t
existence of fixed or coincidence points.

The aim in this paper is to unify the main results of [5] and [6]. Consequently, we obtain very
type of best approximation theorems which can be applied to fixed point theorems of the above-me
types. Our basic tool is the Allen type variational inequality due to the second author [4], which

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us our arguments much shorter than [5]. Finally, we add a coincidence theorem and some remarks recent work of Ding and Tarafdar [3].

2. PRELIMINARIES

A convex space X is a nonempty convex set with any topology that induces the Euclidean topoloty convex hulls of its finite subsets. For such spaces, we have the following version of the Allen type vari inequality due to the second author [4, Theorem 0]:

Theorem 0. *Let X be a convex space, $\psi : X \times X \to \mathbb{R}$ a real function, and K a nonempty compac of X. suppose that*

(0.1) $\psi(x, x) \leq$ *for all* $x \in X$*;*

(0.2) *for each* $y \in X, \{x \in X : \psi(x, y) > 0\}$ *is compactly open;*

(0.3) *for each* $x \in X, \{y \in X : \psi(x, y) > 0\}$ *is convex or empty; and*

(0.4) *for each* $N \in \langle X \rangle$*, there exists a compact convex subset* L_N *of* X *containing* N *such that, for each* $x \in L_N \setminus K$*, there exists a* $y \in L_N$ *satisfying* $\psi(x, y) > 0$*.*

Then there exists an $x_0 \in K$ *such that* $\psi(x_0, y) \leq 0$ *for all* $y \in X$*.*

In Theorem 0, $\langle X \rangle$ denotes the set of all nonempty finite subsets of X.

For a subset X of a vector space E and $x \in E$, the inward set of X at x is defined by

$$I_X(x) := \{x + r(u - x) \in E : u \in X, r > 0\}.$$

Let X be a convex space, E a topological vector space, and $p : X \times E \to \mathbb{R}$ a function which is in the second variable. A function $g : X \to E$ is said to be almost p-quasiconvex [5] or p-affine [6] if $x, x_1, x_2 \in X, z \in E$, and $r \in [0, 1]$, we have

$$p[x, z - g(rx_1 + (1 - r)x_2) \leq \max\{p(x, z - gx_i) : i = 1, 2\}.$$

3. MAIN RESULTS

The following Browder type best approximation theorem is our main result:

Theorem 1. *Let X be a convex space, K a nonempty compact subset of X, E a topolotical vecto* $p : X \times E \to \mathbb{R}$ *a continuous map, and* $f, g : X \to E$ *continuous maps. Suppose that*

(1.1) *for each* $x \in X, p(x, \cdot)$ *is a convex function on* E*;*

(1.2) *g is almost p-quasiconvex; and*

(1.3) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \setminus K$, there exists a $y \in L_N$ such that*

$$p(x, fx - gx) > p(x, fy - gx).$$

Then there exists an $u \in K$ such that

$$p(u, fu - gu) = \inf\{p(u, fu - gu) : x \in X\}.$$

Further, if $g(X)$ is convex, then

$$p(u, fu - gu) = \inf\{p(u, fu - z) : z \in ClI_{g(X)}(gu)\}.$$

Proof. Define $\psi : X \times X \to \mathbb{R}$ by

$$\psi(x, y) = p(x, fx - gx) - p(x, fx - gy)$$

for $(x, y) \in X \times X$ and use Theorem 0. Then

(0.1) $\psi(x, x) = 0$ for all $x \in X$;

(0.2) for each $y \in X$, $\{x \in X : p(x, fx - gx) > p(x, fx - gy)\}$ is open since $x \mapsto \psi(x, y)$ is continu

(0.3) for each $x \in X$, $\{y \in X : p(x, fx - gx) > p(x, fx - gy)\}$ is convex; in fact, for any y, and y set and $0 < r < 1$, we have

$$\begin{aligned} p(x, fx - g(ry_1 + (1-r)y_2)) &\leq \max\{p(x, fx - gy_i) : \ i = 1, 2\} \\ &< p(x, fx - gx) \end{aligned}$$

by (1.2); and

(0.4) clearly follows from (1.3).

Therefore, by Theorem 0, there exists a $u \in K$ such that $\psi(u, y) = p(u, fu - gu) - p(u, fu - gy) \leq$ for all $y \in X$; that is,

$$p(u, fu - gu) = \inf\{p(u, fu - gy) : \ y \in X\}.$$

Further, if $g(X)$ is convex, then for any $z \in I_{g(X)}(gu) \setminus g(X)$, there exist $w \in g(X)$ and $r > 1$ su $z = gu + r(w - gu)$. Suppose that $p(u, fu - gu) > p(u, fu - z)$. Since

$$w = \frac{1}{r}z + (1 - \frac{1}{r})gu \in g(X)$$

and p is convex (in the second variable) by (1.1), we have

$$p(u, fu - w) \leq \frac{1}{r} p(u, fu - z) + (1 - \frac{1}{r}) p(u, fu - gu)$$
$$< p(u, fu - gu),$$

a contradiction. Therefore, $p(u, fu-gu) \leq p(u, fu-z)$ for all $z \in I_{g(X)}(gu)$ and hence for all $z \in clI_{g(}$
Since $gu \in clI_{g(X)}(gu)$, we have

$$p(u, fu - gu) = \inf\{p(u, fu - z) : z \in clI_{g(X)}(gu)\}.$$

This completes our proof.

Remarks 1. 1. In certain case, the inward set in the conclusion can be replaced by the outward s
[1] and [2].

2. The continuities of f, g, and p are used only to assure the compactly openness of $\{x \in X : \psi(x,$
and that $p(u, fu - gu) \leq p(u, fu - z)$ holds for all $z \in clI_{g(X)}(gu)$.

The following is a variation of Theorem 1:

Theorem 2. *Under the hypothesis of Theorem 1, let X be a subset of E. If $gu \in X$, then we hav*

$$p(u, fu - gu) = \inf(p(u, fu - z) : z \in clI_X(gu)\}.$$

Proof. If $gu \in X$, then for any $z \in I_X(gu) \setminus X$, there exist $w \in X$ and $r > 1$ such that $z = gu + r(u$
Therefore, as in the last part of the proof of Theorem 1, we have $p(u, fu-gu) \leq p(u, fu-z)$ for all $z \in$
and hence for all $z \in clI_X(gu)$. This completes our proof.

Remarks 1. Note that, in Theorem 2, X may not have the relative topology with respect to E.

2. Theorem 2 generaliges the main result of Sehgal and Singh [6, Theorem 5], where X is a
subset of a locally convex topological vector space E, and $g : X \to X$. Moreover, they adopted mor
coercivity condition than (1.3).

3. As in [6], Theorem 2 can be applied to obtain some general forms of known results due to Fan, B
and Sehgal-Singh-Gastl.

4. FOR SEMINORMED SPACE ONLY

Theorem 1 can be applied to seminormed vector spaces. In fact, the following is a general Pro
best approximation theorem due to the second author [5, Theorem 2].

Theorem 3. *Let X be a convex space, K a nonempty compact subset of $X, (E, q)$ a seminormec space, where $q : E \to [0, \infty)$ is a seminorm and $f, g : X \to (E, q)$ continuous maps. Suppose that*

(3.1) *g is almost q-guasiconvex; and*

(3.2) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, f $x \in L_N \setminus K$, there exists a $y \in L_N$ satisfying*

$$q(fx - gx) > q(fx - gy).$$

Then there exists a $u \in K$ such that

$$q(fu - gu) = \inf\{q(fu - gx) : x \in X\}.$$

Further if $g(X)$ is convex, then

$$q(fu - gu) = \inf\{q(fu - z) : z \in clI_{g(X)}(gu)\}.$$

In this case, $gu \in Bdg(X)$ and $fu \notin clI_{g(X)}(gu)$ if $q(fu - gu) > 0$.

Proof. Define $p : X \times E \to [0, \infty)$ by $p(x, z) = q(z)$ for $(x, z) \in X \times E$. Then the conclusion follow Theorem 1. Note that if $gu \in Intg(X)$, then $I_{g(X)}(gu) = E$ and $fu \in E$, whence we have $q(fu - g$ This completes our proof.

Remarks 1. In [5], it is shown that Theorem 3 includes earlier works of Fan, Prolla, Hadzič, Sehgal Smithson, Sessa, Roux-Sing, Sehgal-Singh-Gastl, Carbone, Sessa-Singh, Park, and Lin, even when subset of E.

2. In []5], some equivalent form of Theorem 3 and their applications to coincidence results were g

The following is a variation of Theorem 3:

Theorem 4. *Under the hypothesis of Theorem 3, let X be a subset of E. If $gu \in X$, then we hav*

$$q(fu - gu) = \inf\{q(fu - z) : z \in clI_X(gu)\}.$$

In this case, $gu \in BdX$ and $fu \notin clI_X(gu)$ if $q(fu - gu) > 0$.

Remarks 1. Note that, even when X is a subset of E, X may not have the relative topology.

2. Theorem 4 also has a lot of known particular forms as for Theorem 3.

5. COINCIDENCE THEOREMS

Let (E, τ) be a topological vector space, E^* its topological dual, w the weak topology of E, and S

the family of all continuous seminorms on (E, w).

From Theorem 3, we can deduce the following by simply modifying the proof of [5, Theorem 7]:

Theorem 5. *Let X be a convex space, K a nonempty compact subset of X, and (E, τ) a topologica space on which E^* separates points. Let $f, g : X \to (E, w)$ be continuous maps such that*

(7.1) g is almost q-quasiconvex for each $q \in S(E, w)$; and

(7.2) for each $q \in S(E, w)$ and each $N \in \langle X \rangle$, there exists a compact convex subset $L_N \setminus K$, ther a $y \in L_N$ satisfying $q(fx - gx) > q(fx - gy)$.

Then either

(i) f and g have a coincidence point $u \in K$, or

(ii) there exist a $q \in S(E, w)$ and a $u \in K$ such that $gu \in Bdg(X)$ and

$$0 < q(fu - gu) = \inf\{q(fu - z) : z \in g(X)\}.$$

Further if $g(X)$ is convex, the above inequality in (ii) can be uplaced by

$$0 < q(fu - gu) = \inf\{q(fu - z) : z \in clI_{g(X)}(gu)\}.$$

Remarks 1. Theorem 5 is a slightly different version of [5, Theorem 7] and a sharpened form of D Tarafdar [3, Theorem 3.1].

2. From Theorem 5, we can deduce coincidence theorems like [5, Theorem 8], which general Theorem 3.2].

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