BEST APPROXIMATIONS AND FIXED POINTS OF NONEXPANSIVE MAPS IN HILBERT SPACES

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Abstract

We obtain best approximation and fixed point theorems on nonexpansive maps f from a closed convex (not necessarily bounded) subset into a whole Hilbert space. Many boundary conditions for existence of fixed points are collected. Finally, we construct approximating sequences to those fixed points.

1. Introduction

In this paper, we give best approximations and fixed point theorems on nonexpansive maps f from a closed convex (not necessarily bounded) subset into a whole Hilbert space. Those maps f can have many types of boundary conditions such as weakly inwardness or the Leray-Schauder condition in order to ensure existence of fixed points. We also construct approximating sequences to those fixed points.

Consequently, we extend, unify, and improve many known results on nonexpansive maps in Hilbert spaces due to Browder [B1-3], Singh and Watson [SW1,2], Lin and Yen [LY], and many others.

We need the following particular form of Park [P2, Theorem 4]:

Theorem A. *Let X be a uniformly convex Banach space, K a nonempty closed convex subset of X, and $f : K \to X$ a nonexpansive map such that K is (KR)-bounded; that is, for some bounded set $A \subset K$ the set*

$$G(A) = \bigcap_{a \in A} G(a, fa)$$

is either empty or bounded, where

$$G(a, fa) = \{z \in K : \|z - a\| \geq \|z - fa\|\}.$$

Suppose that one of the following holds:

(a) *f is weakly inward; that is, $fx \in \overline{I}_K(x)$.*

(b) *$0 \in \operatorname{Int} K$ and $fx \neq \lambda x$ for all $x \in \operatorname{Bd} K$ and $\lambda > 1$.*

Then f has a fixed point.

Note that Theorem A(a) is due to Kirk and Ray [KR, Theorem 2.2].

Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, respectively. The inward set $I_K(x)$ of K at $x \in X$ is defined by

$$I_K(x) = \{x + r(y - x) : y \in K \quad \text{and} \quad r \geq 0\}.$$

Now we introduce one more fixed point theorem of Kirk and Morales [KM, Theorem 3]:

Theorem B. *Let K be a bounded closed convex subset of a uniformly convex Banach space X, and $f : K \to X$ a nonexpansive map satisfying*

(KM) $0 \in \operatorname{Int} K$ *and* $\lambda \in \mathcal{E}_{\operatorname{Bd} K}$ *implies* $\mathcal{E}_K \cap (1, \lambda) \neq \emptyset$.

Then f has a fixed point in K.

Here, $\mathcal{E}_H = \{\lambda > 1 : fx = \lambda x \text{ for some } x \in H\}$ for $H \subset K$. Note that the Leray-Schauder condition (b) is equivalent to the assumption $\mathcal{E}_{\operatorname{Bd} K} = \emptyset$, the vacuous case of (KM).

2. Best Approximation Theorems

For a closed convex subset K of a Hilbert space X, the proximity map $p = p_K : X \to K$ is defined by

$$px = \{y \in K : \|x - y\| = d(x, K)\} \quad \text{for} \quad x \in X,$$

where $d(x, K) = \inf_{y \in K} \|x - y\|$. It is well-known that p is well-defined and nonexpansive.

We begin with the following best approximation theorem:

Theorem 1. *Let K be a closed convex subset of a Hilbert space X, $f : K \to X$ a nonexpansive map, and $p : X \to K$ the proximity map. Suppose that one of the following holds:*

(i) *K is bounded.*

(ii) *$f(K)$ is bounded.*

(iii) *There exists a $u \in K$ such that the set*

$$G(u, fu)^* = \{z \in K : \|z - fu\| \leq \|z - u\| + d(fu, K)\}$$

is bounded.

(iv) *There exists a* $u \in K$ *such that the set*

$$LS(u,(pf)u) = \{z \in K : \operatorname{Re}\langle (pf)u - u,\ z - u\rangle > 0\}$$

is bounded.

(v) *There exists a* $u \in K$ *such that the set*

$$G(u,(pf)u) = \{z \in K : \|z - (pf)u\| \le \|z - u\|\}$$

is bounded.

(vi) *There exists a bounded set* $A \subset K$ *such that the set*

$$G(A) = \bigcap_{u \in A} G(u,(pf)u)$$

is either empty or bounded.

Then either f *has a fixed point in* K*; or there exists an* $x_0 \in \operatorname{Bd} K$ *such that*

$$0 < \|x_0 - fx_0\| = d(fx_0, K) = d(fx_0, \overline{I}_K(x_0)).$$

Proof. Note that (i) $\Longrightarrow$ (ii) and, for case (ii), since p and f are nonexpansive, so is $pf : K \to K$. Since $f(K)$ is bounded, so are $pf(K)$ and its convex closure $B \subset K$. Therefore, $pf : B \to B$ has a fixed point $x_0 \in B$ by Browder's theorem [B2]. Hence,

$$\|x_0 - fx_0\| = \|(pf)x_0 - fx_0\| = d(fx_0, K).$$

If $(pf)u = u$ in (iv)-(vi), then $G(u,(pf)u) = G(A) = K$ is bounded and hence any of (iv)-(vi) reduces to (i). Therefore, we may assume that $(pf)u \neq u$ in (iv)-(vi).

Note that (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) and the only nontrival part is (iii) $\Longrightarrow$ (v). This is shown by Marino and Trombetta [MT] as follows: For any $z \in G(u,(pf)u)$, we have

$$(*) \qquad \|z - (pf)u\| \le \|z - u\|.$$

Hence

$$\begin{aligned}\|z - fu\| &\leq \|z - (pf)u\| + \|(pf)u - fu\| \\ &\leq \|z - u\| + d(fu, K).\end{aligned}$$

This shows $G(u,(pf)u) \subset G(u, fu)^*$.

Note also that (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) and the only nontrivial part is (iv) $\Longrightarrow$ (v). In fact, for any $z \in G(u,(pf)u)$, from the inequality

$$\|(pf)u - u\|^2 + \|z - u\|^2 - 2\mathrm{Re}\,\langle (pf)u - u, z - u\rangle = \|(pf)u - z\|^2$$

and $\|z - (pf)u\| \leq \|z - u\|$, we have

$$2\mathrm{Re}\,\langle (pf)u - u, z - u\rangle \geq \|(pf)u - u\|^2 > 0$$

since $(pf)u \neq u$. This shows $G(u,(pf)u) \subset LS(u,(pf)u)$.

For case (vi), by Theorem A, $pf : K \to K$ has a fixed point $x_0 = (pf)x_0 \in K$; that is, $\|x_0 - fx_0\| = d(fx_0, K)$.

Now, for any case (i)-(vi), if x_0 is a fixed point, then we have done. Otherwise, we have

$$0 < \|x_0 - fx_0\| = d(fx_0, K).$$

For any $y \in I_K(x_0)\backslash K$, there exist $x \in K$ and $r > 0$ such that $y = x_0 + r(x - x_0)$. Then we must have $r > 1$, for otherwise, $y = (1-r)x_0 + rx \in K$ since $0 < r \leq 1$ and K is convex. Suppose $\|y - fx_0\| < \|x_0 - fx_0\|$. Since $r > 1$ and

$$\frac{1}{r}y + (1 - \frac{1}{r})x_0 = x \in K,$$

we have

$$\begin{aligned}\|x - fx_0\| &= \|\frac{1}{r}(y - fx_0) + (1 - \frac{1}{r})(x_0 - fx_0)\| \\ &\leq \frac{1}{r}\|y - fx_0\| + (1 - \frac{1}{r})\|x_0 - fx_0\| < \|x_0 - fx_0\|,\end{aligned}$$

which contradicts $\|x_0 - fx_0\| = d(fx_0, K)$. Therefore, we must have $\|x_0 - fx_0\| \le \|y - fx_0\|$ for all $y \in I_K(x_0)$, and hence, for all $y \in \overline{I}_K(x_0)$. Therefore, we have $0 < \|x_0 - fx_0\| = d(fx_0, K) = d(fx_0, \overline{I}_K(x_0))$.

If $x_0 \in \operatorname{Int} K$, it is well-known that $\overline{I}_K(x_0) = X$ and $d(fx_0, \overline{I}_K(x_0)) = 0$. This leads a contradiction. Therefore, $x_0 \in \operatorname{Bd} K$. This completes our proof.

Theorem 1(ii) is due to Singh and Watson [SW1, Theorem 5], and Theorem 1(iii) to Marino and Trombetta [MT, Theorem 1]. In [MT], some particular forms of Theorem 1(iii) and examples are given. The conclusion of Theorem 1 is stronger than that of results in [SW, MT]. Note that if $f : K \to K$, then Theorem 1(iv) reduces to Goebel and Kuczumow [GK, Theorem 2], and Theorem 1(i) to the well-known result of Browder [B2, Theorem 1].

3. Fixed Point Theorems

From Theorem 1, we have the following fixed point results:

Theorem 2. *Let K, X, and f be the same as in Theorem 1. Then f has a fixed point whenever one of the following boundary conditions is satisfied for $x \in \operatorname{Bd} K$ such that $x \neq fx$ (except (10)):*

(1) *There exists a $y \in \overline{I}_K(x)$ satisfying*

$$\|y - fx\| < \|x - fx\|.$$

(2) *There exists a number λ (real or complex, depending on whether X is real or complex) such that $|\lambda| < 1$ and*

$$\lambda x + (1 - \lambda) fx \in \overline{I}_K(x).$$

(3) *$fx \notin p_W^{-1}(x)$, where $p_W : X \to \overline{I}_K(x)$ is the proximity map and $W = \overline{I}_K(x)$.*

(4) *$fx \in \overline{I}_K(x)$.*

(5) *There exists a* $y \in K$ *satisfying*

$$\|y - fx\| < \|x - fx\|.$$

(6) $LS(x, fx) = \{z \in K : \mathrm{Re}\,\langle z - x, fx - x\rangle > 0\} \neq \emptyset.$

(7) $\liminf_{h\to 0^+} d[(1-h)x + h(fx), K]/h < \|x - fx\|.$

(8) $\lim_{h\to 0^+} d[(1-h)x + h(fx), K)/h = 0.$

(9) *There exsits a* $y \in K$ *satisfying*

$$\|fx - y\| \le \|x - y\|.$$

(10) *If* $x = (pf)x$*, then* x *is a fixed point.*

(11) $f(\mathrm{Bd}\,K) \subset K$.

(12) $f(K) \subset K$.

Proof. (1) Condition (1) violates the second alternative of the conclusion of Theorem 1.

(2) Note that $y := \lambda x + (1-\lambda)fx \in \overline{I}_K(x)$ satisfies condition (1). In fact,

$$\|y - fx\| = \|\lambda x - \lambda fx\| = |\lambda|\,\|x - fx| < \|x - fx\|.$$

(3) Since $fx \notin p_W^{-1}(x)$, we have $x \neq (p_W f)x$. Hence, there exists a $y \in W = \overline{I}_K(x)$ satisfying $\|y - fx\| < \|x - fx\|$. Therefore, (1) holds.

(4) Note that $y = fx$ satisfies $\|y - fx\| < \|x - fx\|$. Therefore, (1) holds. Note also that (12) $\Longrightarrow$ (11) $\Longrightarrow$ (4).

(5) Since $y \in K \subset \overline{I}_K(x)$ satisfies $\|y - fx\| < \|x - fx\|$, condition (1) is satisfied.

(6),(7) Williamson [W, Propositions 1 and 2] showed that (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7).

(8) It is known that (8) $\Longleftrightarrow$ (4) for any closed convex subset K of a Banach space. See Caristi [C].

(9) Note that (9) $\Longrightarrow$ (5). See Schöneberg [Sc], Singh and Watson [SW1], or Park [P1].

(10) Follow the proof of Singh and Watson [SW1, Theorem 7].

Theorem 2 includes results due to Browder [B1, B2], Browder and Petryshyn [BP], Lin and Yen [LY], Park [P1], Ray and Cramer [RC], Reich [R], and Singh and Watson [SW1].

From Theorem A, we have the following:

Theorem 3. *Let K be a closed convex subset of a Hilbert space X, and $f : K \to X$ a nonexpansive map such that*

(vii) *K is (KR)-bounded.*

Then f has a fixed point whenever one of the following boundary conditions holds:

(4) *f is weakly inward.*

(8) $\lim_{h \to 0^+} d[(1-h)x + h(fx),\ K]/h = 0$.

(LS) $0 \in \operatorname{Int} K$ *and* $fx \neq \lambda x$ *for all* $x \in \operatorname{Bd} K$ *and* $\lambda > 1$.

(A) $0 \in \operatorname{Int} K$ *and* $\|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2$ *for all* $x \in \operatorname{Bd} K$.

(K) $0 \in \operatorname{Int} K$ *and* $\operatorname{Re} \langle fx, x \rangle \leq \|x\|^2$ *for all* $x \in \operatorname{Bd} K$.

Proof. It is easy to see that (vi) $\Longrightarrow$ (vii) and hence all of conditions (i)-(vi), except (ii), is included in (vii). Therefore, if (4) holds, then f has a fixed point by Theorem A(a), and if (LS) holds, then f has a fixed point by Theorem A(b). Note that (4) $\Longleftrightarrow$ (8) and it is easy to see that (A) $\Longrightarrow$ (LS) and (K) $\Longrightarrow$ (LS). This completes our proof.

Theorem 3 contains results due to Krasnoselskii [Kr] and Shinbrot [Sh].

From Theorems 2, 3, and B, we have the following:

Theorem 4. *Let K be a bounded closed convex subset of a Hilbert space X and $f : K \to X$ a nonexpansive map. Then f has a fixed point whenever one of the boundary conditions* (1)-(12), (KM), (LS), (A), *and* (K) *is satisfied.*

Theorem 4 has a long history and originates from Browder [B1,B2]. For the origin of each boundary condition, see [P1, P2]. Especially, the so-called Rothe condition (11) was actually due to Knaster, Kuratowski, and Mazurkiewicz [KKM] and the so-called Leray-Schauder condition (LS) to Bohl [B].

4. Approximating Fixed Points

In this section, we show that we can construct a sequence of approximating fixed points which converges strongly to a fixed point.

Theorem 5. *Let X be a real Hilbert space, K a closed convex subset of X, $0 \in K$, and $f : K \to X$ a nonexpansive map. Suppose that one of the following holds:*

(a) *K satisfies one of* (i)-(vi) *and f satisfies one of* (1)-(12)*.*

(b) *K is (KR)-bounded and f satisfies one of* (4), (8), (11), (12), (LS), (A), *and* (K)*.*

(c) *K is bounded and f satisfies one of* (1)-(12), (KM), (LS), (A), *and* (K)*.*

Let $f_k x = k(fx) + (1-k)x_0$ for some $x_0 \in K$ and $0 < k < 1$, $k \to 1$, and let $f_k x_k = x_k$. Then x_k converges strongly to y_0, where y_0 is the fixed point of f closest to x_0.

Proof. The fixed point set $\mathrm{Fix}(f)$ of f is nonempty by Theorems 2-4. Then $\mathrm{Fix}(f)$ is a closed convex subset of K [B2]. So there exists a unique closest point $y_0 = fy_0$ to x_0. Now we follow the method of Singh and Watson [SW2, Theorem]. For simplicity we take $x_0 = 0$. Then

$$\|x_k/k - y_0\| = \|fx_k - y_0\| = \|fx_k - fy_0\| \le \|x_k - y_0\|.$$

Hence, $\|x_k/k - y_0\|^2 \le \|x_k - y_0\|^2$ implies

$$\|x_k\|^2 + k^2\|y_0\|^2 - 2k\langle x_k, y_0\rangle \le k^2(\|x_k\|^2 + \|y_0\|^2 - 2\langle x_k, y_0\rangle),$$

or

$$\|x_k\|^2 \le \frac{2k}{1+k}\langle x_k, y_0\rangle \le \langle x_k, y_0\rangle.$$

Therefore,

$$\|x_k\| \le \langle \frac{x_k}{\|x_k\|}, y_0\rangle \le \frac{\|x_k\|}{\|x_k\|}\|y_0\| \quad \text{or} \quad \|x_k\| \le \|y_0\|.$$

Since $\{x_k\}$ is bounded, there is a weakly convergent subsequence $\{x_{k_i}\}$; that is, $x_{k_i} = x_i \rightharpoonup x$ for some x. Then

$$\|x_i - fx_i\| = \|k_i fx_i - fx_i\| = |k_i - 1| \, \|fx_i\| \to 0$$

or $x_i - fx_i \to 0$ $(i \to \infty)$. [$\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.]

Since $f : K \to H$ is nonexpansive, $1 - f$ is demiclosed [B3]; that is, $x_i \rightharpoonup x$ and $x_i - fx_i \to 0$ implies $x - fx = 0$ or $x = fx$. Since $\|x_i\| \leq \|y_0\|$, so $\|x\| \leq \|y_0\|$. Since y_0 is closest to $x_0 = 0$, so $\|x\| = \|y_0\|$ or $x = y_0$, and hence $x_i \rightharpoonup y_0$. Therefore, y_0 is the only weakly cluster point of $\{x_k\}$ as $k \to 1$, and hence $x_k \rightharpoonup y_0$. Now we show that $x_k \to y_0$. Indeed,

$$\begin{aligned} \|y_0\|^2 \geq \|x_k\|^2 &= \|x_k - y_0 + y_0\|^2 \\ &= \|x_k - y_0\|^2 + \|y_0\|^2 + 2\langle x_k - y_0, y_0 \rangle \end{aligned}$$

and $\langle x_k - y_0, y_0 \rangle \to 0$ as $k \to 1$, we have $\|x_k - y_0\| \to 0$. Thus $x_k \to y_0$. This completes our proof.

Browder [B3] obtained Theorem 5 for case (i) and (12), Singh and Watson [SW2] for case (ii) and (11), and Xu and Yin [XY] for case (c) and (4) or (LS).

Finally, some of our results have generalized forms for Banach spaces or locally convex topological vector spaces and for 1-set-contractive maps, for example, see Lin and Yen [LY], Kirk and Morales [KM].

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