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COMMENTS ON TASKOVIĆ'S EXTENSION OF THE BROUWER THEOREM

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ABSTRACT. This is to clarify that Tasković's extension of the Brouwer fixed point theorem is incorrectly stated.

In [2], M. R. Tasković claimed that he obtained an extension of the Brouwer fixed point theorem and its generalizations due to Schauder and many others. His proof, if it were correct, is extremely elementary and constructive. We noticed that his claim was incorrect, but, until now, no one has insisted that. In this paper, we clarify this matter with some remarks.

The following is a restatement of a main result of [2], which is a generalization of the well-known Caristi-Kirk-Browder theorem [1] and the celebrated Banach contraction principle.

THEOREM 4. [2] *Let X be a topological space, $G : X \to [0, \infty)$ a lower semicontinuous function, $A : X \times X \to [0, \infty)$, and $T : X \to X$ a function. Suppose that*

(1) *for any $x \in X$ with $x \neq Tx$, there exists a $y \in X \backslash \{x\}$ such that $A(x, y) \leq G(x) - G(y)$;*
(2) *any sequence $\{a_n\}$ in X satisfying $A(a_n, a_{n+1}) \to 0$ (as $n \to \infty$) has a convergent subsequence; and*
(3) *$y \mapsto A(x, y)$ is lower semicontinuous and $A(x, y) = 0$ implies $x = y$.*

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Then T has a fixed point.

In [2], it is claimed that Theorem 4 extends the well-known fixed point theorems of Brouwer, Schauder, and many others in view of the following:

LEMMA 2. [2] *Let (X, ρ) be a metric space. If C is a metric convex set in X and $T : C \to C$, then there exists a lower semicontinuous function $G : C \to [0, \infty)$ satisfying condition* (1) *of Theorem 4 for $\rho = A$.*

We claim that Lemma 2 is false with the following reasons:

1. In the proof of Lemma 2, Tasković defined

$$G(x) = \begin{cases} 3\rho[a, x] & \text{for } \; x \in C \backslash \{a\} \\ 3\rho[Ta, x] & \text{for } \; x = a. \end{cases}$$

However, G is not lower semicontinuous in general. In fact, consider a sequence $\{x_n\}$ in C such that $x_n \to a$ and $x_n \neq a$. Then $G(x_n) = 3\rho[a, x_n] \to 0$, but $G(a) = 3\rho[Ta, a] \neq 0$ by assumption.

2. If Lemma 2 were true, then any (not necessarily continuous) selfmap T of a closed convex subset of a Banach space would have a fixed point. (This is clearly false.) In fact, suppose that T has no fixed point. Then, by Lemma 2, for any $x \in C$, there exists an $f(x) := y \in C \backslash \{x\}$ satisfying $\|x - f(x)\| \leq G(x) - G(f(x))$. Then by the Caristi-Kirk-Browder theorem or Theorem 4, f has a fixed point, a contradiction. This implies the existence of a fixed point of T.

This is why Tasković claimed that his Theorem 4 is an extension of the Brouwer fixed point theorem and many of its generalizations due to Schauder and many others.

Finally, note that there have appeared many generalizations of the Caristi-Kirk-Browder theorem which are similar to Theorems 1-4 of [2].

References

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