FIVE EPISODES RELATED TO GENERALIZED CONVEX SPACES

SEHIE PARK

Abstract. Certain results on G-convex spaces are used to obtain improved versions of recent works on the Fan-Browder type coincidence theorems [TZ, LL], existence of maximal elements [TZ], the Fan type minimax theorems [TZ], the G-KKM family and fixed points [P1, LL], and topological semilattices [HC].

0. Introduction and preliminaries

Recently, we introduced the concept of generalized convex (or G-convex) spaces as a far-reaching generalization of convex spaces, H-spaces, and other convex structures. We established in such a context the foundations of the KKM theory, as well as fixed point theorems and other results for multimaps. See Park and Kim [PK1-6]. This direction of study was immediately followed by Tan *et al.* [CT, T, TZ].

Our aim in the present paper is to show that our theory can be used to obtain improved versions of recent works of others: The Fan-Browder type coincidence theorems [TZ, LL], existence of maximal elements [TZ], the Fan type minimax theorems [TZ], the G-KKM family and fixed points [P1, LL], and results on topological semilattices [HC].

A *multimap* (or *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. For any $B \subset Y$, the *lower inverse* of B under F is defined by

$$F^{-}(B) = \{x \in X : Fx \cap B \neq \emptyset\}.$$

The (*lower*) *inverse* of $F : X \multimap Y$ is the map $F^- : Y \multimap X$ defined by $x \in F^- y$ if and only if $y \in Fx$. Given two maps $F : X \multimap Y$ and $G : Y \multimap Z$, the *composite* $GF : X \multimap Z$ is defined by $(GF)x = G(Fx)$ for $x \in X$.

For topological spaces X and Y, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X.

An *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is a class such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma x \subset Tx$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and nonempty compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Note that a polytope is a convex hull of a nonempty finite subset of a vector space with the Euclidean topology.

For details on admissible maps, see [P2, PK1].

For a nonempty set D, let $\langle D \rangle$ denote the set of all nonempty finite subsets of D. For a set A, let $|A|$ denote the cardinality of A. Let Δ_n denote the standard n-simplex; that is,

$$\Delta_n = \{u \in \mathbf{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u) e_i, \ \lambda_i(u) \geq 0, \ \sum_{i=1}^{n+1} \lambda_i(u) = 1\},$$

where e_i is the i-th unit vector in $\mathbf{R}^{n+1}$.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X, and a function $\Gamma : \langle D \rangle \multimap X \backslash \{\emptyset\}$ such that

(1) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and
(2) for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$. For details on G-convex spaces, see [PK1,2].

We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D \rangle$. For an $(X, D; \Gamma)$, a subset C of X is said to be *G-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

For a nonempty subset A of X, the *G-convex hull* of A, G-co A, is defined by

$$G\text{-co}\, A = \bigcap \{Y : A \subset Y \subset X \text{ and } Y \text{ is } G\text{-convex}\}.$$

The following is known in [PK3]:

Lemma 1. *Let $(X, D; \Gamma)$ be a G-convex space, and $R : D \multimap X$ such that*

(1) *for each $x \in D$, Rx is closed [resp., open]; and*

(2) *for each $N \in \langle D \rangle$, $\Gamma_N \subset R(N)$.*

Then $\{Rx : x \in D\}$ has the finite intersection property.

The following KKM theorem for G-convex spaces is due to Park and Kim [PK2, Theorem 3]:

Theorem A. *Let (X, D, Γ) be a G-convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. Let $R : D \multimap Y$ be a multimap such that*

(A.1) *for each $x \in D$, Rx is compactly closed in Y;*

(A.2) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset R(N)$; and*

(A.3) *there exist a nonempty compact subset K of Y such that either*

(i) *$\bigcap \{Rx : x \in M\} \subset K$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, a compact G-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap \{Rx : x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap \{Rx : x \in D\} \neq \emptyset$.

Note that if F is single-valued, we do not need the Hausdorffness of Y.

1. The Fan-Browder type coincidence theorems

We begin with the following coincidence theorem due to Park and Kim [PK2, Theorem 1]:

Theorem B. *Let* $(X, D; \Gamma)$ *be a* G*-convex space,* Y *a Hausdorff space,* $S : D \multimap Y$, $T : X \multimap Y$ *maps, and* $F \in \mathfrak{A}_c^\kappa(X, Y)$*. Suppose that*

(B.1) *for each* $x \in D$, Sx *is compactly open in* Y*;*

(B.2) *for each* $y \in F(X)$, $M \in \langle S^- y \rangle$ *implies* $\Gamma_M \subset T^- y$*;*

(B.3) *there exists a nonempty compact subset* K *of* Y *such that* $\overline{F(X)} \cap K \subset S(D)$*; and*

(B.4) *either*

(i) $Y \backslash K \subset S(M)$ *for some* $M \in \langle D \rangle$*; or*

(ii) *for each* $N \in \langle D \rangle$*, there exists a compact* G*-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \backslash K \subset S(L_N \cap D)$*.*

Then there exists an $\overline{x} \in X$ *such that* $F\overline{x} \cap T\overline{x} \neq \emptyset$*.*

In was observed that if F is single-valued, then we do not need the Hausdorffness of Y in Theorem B.

If $X = D = Y$ and $F = 1_X$, then Theorem B reduces to the following generalization of the Fan-Browder fixed point theorem.

Theorem 1. *Let* (X, Γ) *be a* G*-convex space and* $S, T : X \multimap X$ *maps such that*

(1.1) *for each* $x \in X$, Sx *is compactly open in* X*;*

(1.2) *for each* $y \in X$, $M \in \langle S^- y \rangle$ *implies* $\Gamma_M \subset T^- y$*;*

(1.3) *there exists a nonempty compact subset* K *of* $S(X)$*; and*

(1.4) *either*

(i) $X \backslash K \subset S(M)$ *for some* $M \in \langle X \rangle$*; or*

(ii) *for each* $N \in \langle X \rangle$*, there exists a compact* G*-convex subset* L_N *of* X *containing* N *such that* $L_N \backslash K \subset S(L_N)$*.*

Then there exists an $\overline{x} \in X$ *such that* $\overline{x} \in T\overline{x}$*.*

Remarks. 1. Condition (1.2) can be replaced by the following:

(1.2)′ $Sx \subset Tx$ for all $x \in X$ and $T^{-}y$ is G-convex for each $y \in X$.

In this case, Theorem 1 with $X = K$ reduces to Tan and Zhang [TZ, Theorem 3.1], which is motivated by [PK2].

2. Tan and Zhang [TZ, Theorems 3.2-3.5] are trivial equivalent formulations of [TZ, Theorem 3.1] and also can be improved as in Theorems B or 1.

3. For an H-space $(X;\Gamma)$, Theorem 1 with $S = T$ and $X = K$ reduces to Lee and Lee [LL, Theorem 2.3].

Let us consider the following selection theorem:

Theorem 2. *Let $(X, D;\Gamma)$ be a G-convex space, Y a Hausdorff compact space, and $S : D \multimap Y$, $T : X \multimap Y$ maps. Suppose that*

(2.1) *for each $y \in Y$, $M \in \langle S^{-}y\rangle$ implies $\Gamma_M \subset T^{-}y$; and*

(2.2) $Y = \bigcup\{\operatorname{Int} Sx : x \in D\}$.

Then $T^{-} : Y \multimap X$ has a continuous selection. More precisely, there exists a simplex Δ_n and two continuous functions $p : Y \to \Delta_n$ and $\phi : \Delta_n \to X$ such that, $(\phi p)y \in T^{-}y$ for all $y \in Y$.

Proof. Since Y is compact, by (2.2), there exists an $N = \{x_1, \ldots, x_{n+1}\} \in \langle D\rangle$ such that $Y = \bigcup_{i=1}^{n+1} \operatorname{Int} Sx_i$. Then there exists a continuous map $\phi_N : \Delta_n \to X$ such that $\phi_N(\Delta_n) \subset \Gamma_N$ and $\phi_N(\Delta_J) \subset \Gamma_J$ for each $J \in \langle N\rangle$. Let $\{\lambda_i\}_{i=1}^{n+1}$ be the partition of unity subordinated to the cover $\{\operatorname{Int} Sx_i\}_{i=1}^{n+1}$ of the Hausdorff compact space Y. Define a continuous function $p : Y \to \Delta_n$ by

$$p(y) = \sum_{i=1}^{n+1} \lambda_i(y)e_i = \sum_{x_i \in N_y} \lambda_i(y)e_i \quad \text{for} \quad y \in Y,$$

where $x_i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in \operatorname{Int} Sx_i \implies x_i \in S^{-}y$ and hence $N_y \in \langle S^{-}y\rangle$. By (2.1), we have $(\phi_N p)y \in \phi_N(\Delta_{N_y}) \subset \Gamma_{N_y} \subset T^{-}y$ for each $y \in Y$. Therefore, $\phi_N p : Y \to X$ is a continuous selection of T^{-}. This completes our proof.

Remarks. 1. Actually, Theorem 2 appears in the proof of Theorem B in [PK2].

2. Particular forms of Theorem 2 are due to Tan and Zhang [TZ, Theorems 2.3 and 2.4], which were used to prove [TZ, Theorem 3.1].

From Theroem 2, we have the following:

Corollary. *Let $(X;\Gamma)$ be a Hausdorff compact G-convex space and $R : X \multimap X$ a map such that*

(1) *for each $x \in X$, Rx is G-convex; and*

(2) $X = \bigcup\{\operatorname{Int} R^{-}y : y \in X\}$.

Then there exists an $\overline{x} \in X$ such that $\overline{x} \in R\overline{x}$.

Proof. From Theorem 2 with $X = D = Y$ and $R = S^{-} = T^{-}$, there exists two continuous functions $p : X \to \Delta_n$ and $\phi : \Delta_n \to X$ such that $(\phi p)x \in Rx$ for $x \in X$. Then by the Brouwer fixed point theorem, $p\phi$ has a fixed point $z \in \Delta_n$; that is, $(p\phi)z = z$. Let $\overline{x} = \phi z$. Then $(\phi p)\overline{x} = \phi(p\phi)z = \phi z = \overline{x}$ and hence $\overline{x} \in R\overline{x}$. This completes our proof.

2. Existence of maximal elements

Any binary relation R in a set X can be regarded as a multimap $T : X \multimap X$ and conversely by the following obvious way:

$$y \in Tx \text{ if and only if } (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a multimap $T : X \multimap X$ if $Tx_0 = \emptyset$.

From Theorem B, we have the following existence theorem for maximal elements:

Theorem 3. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that* (B.1), (B.2), *and* (B.4) *holds for some nonempty compact subset of K. If $Fx \cap Tx = \emptyset$ for all $x \in X$, then there exists a $y \in \overline{F(X)} \cap K$ such that $S^- y = \emptyset$.*

Proof. Suppose that for each $y \in \overline{F(X)} \cap K$, there exists an $x \in S^{-1} y \subset D$. Then $\overline{F(X)} \cap K \subset S(D)$; that is, (1.3) holds. Therefore, by Theorem B, there exists an $\overline{x} \in X$ such that $F\overline{x} \cap T\overline{x} \neq \emptyset$, a contradiction.

Theorems B and 3 are actually equivalent:

Proof of Theorem B using Theorem 3. Suppose that $Fx \cap Tx = \emptyset$ for all $x \in X$, then by Theorem 3, there exists a $y \in \overline{F(X)} \cap K$ such that $S^- y = \emptyset$, and hence $\overline{F(X)} \cap K \not\subset S(D)$. This contradicts (B.3).

Remarks. 1. If $X = D$ is a convex space, Theorem 3 reduces to Park [P3, Theorem 3.1], which includes earlier works of Yannelis and Prabhakar, Mehta, Kim, and Mehta and Sessa.

2. For $X = D = K$ and $F = 1_X$, Theorem 3 reduces to Tan and Zhang [TZ, Theorem 3.5] as follows:

Corollary. *Let $(X; \Gamma)$ be a compact G-convex space and $S : X \multimap X$ be such that for each $y \in X$, $S^- y$ is open in X and for each $x \in X$, $x \notin G$-$\mathrm{co}(Sx)$. Then there exists a $\overline{y} \in X$ such that $S\overline{y} = \emptyset$.*

Remark. In [TZ], the authors applied Corollary to existence of equilibrium points of generalized games.

3. The Fan type minimax theorems

The following is due to Park and Kim [PK3, Theorem 8]:

Theorem C. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, K a nonempty compact subset of Y, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $\phi : D \times Y \to \overline{\mathbf{R}}$ be an extended real-valued function and $\gamma \in \overline{\mathbf{R}}$ such that*

(C.1) *for each $x \in D$, $\{y \in Y : \phi(x, y) \leq \gamma\}$ is compactly closed;*

(C.2) *for each $N \in \langle D \rangle$, and $y \in F(\Gamma_N)$, $\min\{\phi(x, y) : x \in N\} \leq \gamma$; and*

(C.3) *either*

(i) *there exists an $M \in \langle D \rangle$ such that for each $y \in Y \backslash K$, $\phi(x, y) > \gamma$ for some $x \in M$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact G-convex subset L_N of X containing N such that, for each $y \in F(L_N) \backslash K$, there exists an $x \in L_N \cap D$ satisfying $\phi(x, y) > \gamma$.*

Then (a) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that*

$$\phi(x, \hat{y}) \leq \gamma \quad \textit{for all} \quad x \in D; \quad \textit{and}$$

(b) *if $\gamma = \sup\{\phi(x, y) : (x, y) \in F\}$, then we have the minimax inequality:*

$$\min_{y \in K} \sup_{x \in D} \phi(x, y) \leq \sup_{(x,y) \in F} \phi(x, y).$$

Note that if F is single-valued, the Hausdorffness of Y is not necessary as for Theorem B.

If $X = D = Y = K$ and $F = 1_X$, then Theorem C reduces to the following:

Theorem 4. *Let $(X; \Gamma)$ be a compact G-convex space, $\phi : X \times X \to \overline{\mathbf{R}}$ an extended real function, and $\gamma \in \overline{\mathbf{R}}$ such that*

(1) *for each $x \in X$, $\langle y \in X : \phi(x, y) \leq \gamma\}$ is closed; and*

(2) *for each $N \in \langle X \rangle$ and $y \in \Gamma_N$, $\min\{\phi(x, y) : x \in N\} \leq \gamma$.*

Then (a) *there exists a $\overline{y} \in X$ such that*

$$\phi(x, \overline{y}) \leq \gamma \quad \textit{for all } x \in X; \quad \textit{and}$$

(b) *if $\gamma = \sup_{x \in X} \phi(x, x)$, then we have*

$$\min_{y \in X} \sup_{x \in X} \phi(x, y) \leq \sup_{x \in X} \phi(x, x).$$

Remarks. 1. Theorem 4 improves Tan and Zhang [TZ, Theorem 3.6] in several aspects.

2. Similarly, [PK3, Theorem 6] is a far-reaching general form of [TZ, Theorems 3.7 and 3.8].

3. In [TZ], it was shown that Theorems 3.1-3.8 are all equivalent. Some of the equivalencies were already shown in [P2] for convex spaces and in [PK3] for G-convex spaces.

4. The G-KKM family and fixed points

For a G-convex space $(X, D; \Gamma)$, let $F : D \multimap X$ be a multimap. Then $\{Fx : x \in D\}$ is called a *G-KKM family* if $\Gamma_A \subset F(A)$ for any $A \in \langle D \rangle$. In this case, F is called a *G-KKM map.*

We have a fixed point theorem:

Theorem 5. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ multimaps such that*

(5.1) *for each $y \in X$, $M \in \langle S^- y \rangle$ implies $\Gamma_M \subset T^- y$; and*

(5.2) *$\{X \backslash Sx : x \in D\}$ is not a G-KKM family.*

Then there exists an $x_0 \in X$ such that $x_0 \in Tx_0$.

Proof. Let $Fx = X \backslash Sx$ for $x \in X$. Since $F : D \multimap X$ is not a G-KKM map by (5.2), there exists an $M \in \langle D \rangle$ such that $\Gamma_M \not\subset \bigcup\{X \backslash Sx : x \in M\}$. Hence there exists an $x_0 \in \Gamma_M$ such that $x_0 \in Sx$ for all $x \in M$; that is, $M \in \langle S^- x_0 \rangle$. Therefore, $\Gamma_M \subset T^- x_0$ by (5.1), and hence $x_0 \in T^- x_0$ or $x_0 \in Tx_0$. This completes our proof.

Remarks. 1. If $Sx \subset Tx$ for all $x \in D$ and $T^- y$ is G-convex for all $y \in X$, then (5.1) holds immediately. In this case, Theorem 5 with $X = D$ is due to Park [P1, Theorem 4] for convex spaces and to Lee and Lee [LL, Theorem 2.1] for H-spaces.

2. Similarly, Lee and Lee [LL, Corollaries 2.1-2.3] can be extended to G-convex spaces as follows:

Corollary 1. *In Theorem 5, condition (5.2) can be replaced by any of the following without affecting its conclusion:*

(5.2)$'$ *Sx is closed [resp., open] for each $x \in D$, and $X = S(A)$ for some $A \in \langle D \rangle$.*

(5.2)$''$ *X is compact, Sx is open for each $x \in D$, and $S^- y$ is nonempty for each $y \in X$.*

Proof. For the first case, $Fx = X \backslash Sx$ is open [resp., closed] for each $x \in D$ and $\bigcap\{Fx : x \in A\} = \emptyset$. Therefore, by Lemma 1, $\{Fx\}_{x \in D}$ is not a G-KKM family. Therefore, (5.2)$'$ implies (5.2).

For the second case, since $S^- y$ is nonempty for each $y \in X$, we have $y \in Fx$ for some $x \in S^- y \subset D$. Since X is compact and covered by $\{Sx\}_{x \in D}$, X is covered by some $\{Sx_i\}_{i=1}^n$ with $x_i \in D$. Then $\bigcap_{i=1}^n (X \backslash Sx_i) = X \backslash \bigcup_{i=1}^n Sx_i = \emptyset$. Therefore, by Lemma 1, $\{X \backslash Sx : x \in D\}$ is not G-KKM family, and hence (5.2)$''$ implies (5.2). This completes our proof.

Remark. Note that Tan and Zhang [TZ, Theorems 3.1 and 3.3] are simple consequences of Corollary 1.

Corollary 2. *Let $(X, D; \Gamma)$ be a G-convex space, and $T : X \multimap X$ a map such that*

(1) *Tx is G-convex for each $x \in X$;*
(2) *$T^- y$ is closed [resp., open] for each $y \in X$; and*
(3) *there exists an $A \in \langle D \rangle$ such that $Tx \cap A \neq \emptyset$ for each $x \in X$.*

Then T has a fixed point.

Proof. Let $S : D \multimap X$ be defined by $Sx = T^- x$ for $x \in D$. Then (5.1) holds with T instead of T^-; that is, for each $y \in X$, $S^- y \subset Ty$ and Ty is G-convex by (1). Moreover, for each $x \in D$, Sx is closed [resp., open] by (2), and for each $x \in X$, there exists an $a \in A$ such that $a \in Tx$ or $x \in T^- a = Sa$ by (3). Hence, X is covered by a finite number of Sa's, and so (5.2)$'$ holds. Therefore, by Corollary 1, there exists an $x_0 \in X$ such that $x_0 \in T^- x_0$ or $x_0 \in Tx_0$. This completes our proof.

Remarks. 1. Corollary 2 with $X = D$ reduces to Lee and Lee [LL, Corollary 2.4] for H-spaces.

2. From Corollary 2, we have the Fan-Browder theorem for G-convex spaces as follows:

Corollary 3. *Let $(X;\Gamma)$ be a compact G-convex space and $T : X \multimap X$ a map such that*

(1) *Tx is nonempty and G-convex for each $x \in X$; and*

(2) *$T^{-}y$ is open for each $y \in X$.*

Then T has a fixed point.

Proof. For each $x \in X$, there exists a $y \in Tx$ or $x \in T^{-}y$. Since X is compact, $X = \bigcup_{a\in A} T^{-}a$ for some $A \in \langle X \rangle$. Therefore all of the requirements of Corollary 2 are satisfied.

Finally, in this section, we show that the converse of Theorem 5 holds under certain restrictions:

Theorem 6. *Let $(X;\Gamma)$ be a G-convex space, and $T : X \multimap X$ multimaps such that*

(6.1) *for each $y \in X$, $T^{-}y$ is G-convex; and*

(6.2) *$x \in \Gamma_{\{x\}}$ for each $x \in X$.*

Then T has a fixed point if and only if $\{X \backslash Tx : x \in X\}$ is not a G-KKM family.

Proof. Suppose that the map $F : X \multimap X$ defined by $Fx = X\backslash Tx$ for $x \in X$ is a G-KKM map. Then for each $x \in X$, we have

$$x \in \Gamma_{\{x\}} \subset Fx = X\backslash Tx \quad \text{or} \quad x \notin Tx.$$

The converse follows from Theorem 5 with $X = D$ and $S = T$. This completes our proof.

Remark. Theorem 6 reduces to Park [P1, Theorem 7] for convex spaces and to Lee and Lee [LL, Theorem 2.2] for H-spaces.

5. Topological semilattices

Recently, Horvath and Ciscar [HC] studied topological semilattices and established in such a context an order theoretical version of the classical KKM theorem, as well as fixed point theorems for multimaps.

In this section, we show that the contents of the mathematical framework of [HC, Section 2] and their consequences follow from our results on G-convex spaces.

A *semilattice* or, more exactly, a sup-*semilattice*, is a partially ordered set $(X, \leq)$ for which any pair (x, x') of elements has a lub $x \vee x'$. Any $A \in \langle X \rangle$ has a lub denoted by $\sup A$. If $x \leq x'$, then the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called an *order interval.* For details, see [HC].

Lemma 2. *Any topological semilattice $(X, \leq)$ with path-connected intervals is a G-convex space. More precisely, let D be a nonempty subset of X and $\Gamma : \langle D \rangle \multimap X$ a map such that*

$$\Gamma_A = \Gamma(A) = \bigcup_{a \in A} [a, \sup A] \quad \textit{for } A \in \langle D \rangle.$$

Then $(X, D; \Gamma)$ is a G-convex space.

Proof. Note that Γ is well-defined and that

(a) $A \subset \Gamma_A$ for $A \in \langle D \rangle$; and

(b) $A \subset B$ implies $\Gamma_A \subset \Gamma_B$ for $A, B \in \langle D \rangle$.

Moreover

(c) for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. See the proof of [HC, Theorem 1].

Therefore, $(X, D; \Gamma)$ is a G-convex space.

Now we have an order theoretical version of the well-known KKM theorem:

Theorem 7. *Let X be a topological semilattice with path-connected intervals, D a nonempty subset of X, and $R : D \multimap X$ a map such that*

(7.1) *for each $x \in D$, Rx is closed [resp., open] in X; and*

(7.2) *for each $N \in \langle D \rangle$, $\Gamma_N \subset R(N)$.*

Then $\{Rx : x \in D\}$ has the finite intersection property.

Proof. By Lemma 2, $(X, D; \Gamma)$ is a G-convex space. Then we can apply Lemma 1 and have the conclusion immediately.

Remark. Theorem 7 is a restatement of [HC, Theorems 1 and 1′]. Moreover, [HC, Theorem 2] can be restated as follows:

Theorem 8. *Let X be a topological semilattice with path-connected intervals, D a nonempty subset of X, and $R : D \multimap X$ a map such that*

(8.1) *for each $x \in D$, Rx is compactly closed in X;*

(8.2) *for each $N \in \langle D \rangle$, $\Gamma_N \subset R(N)$; and*

(8.3) *Rx_0 is compact for some $x_0 \in D$.*

Then $\bigcap_{x \in D} Rx \neq \emptyset$.

Proof. Since X is a G-convex space by Lemma 2, we can apply Theorem A. Note that condition (8.3) is the simplest case of the coercivity condition (A.3).

The following existence result of continuous selections in [HC, Theorem 3] also follows from our result:

Theorem 9. *Let K be a compact topological space, X a topological semilattice with path-connected intervals, and $R : K \multimap X$ a map such that*

(9.1) *for each $y \in K$, if $x_1, x_2 \in Ry$ then $[x_1, x_1 \vee x_2] \subset Ry$; and*

(9.2) $K = \bigcup \{\operatorname{Int} R^- x : x \in X\}$.

Then there exist a simplex Δ_n *and two continuous functions* $g : \Delta_n \to X$ *and* $h : K \to \Delta_n$ *such that* $(gh)y \in Ry$ *for all* $y \in K$.

Proof. We use Theorem 2 with $X = D$, $Y = K$, and $R = T^- = S^-$. Note that (9.1) immediately implies condition (2.1). Therefore, we have the conclusion by Theorem 2.

Note that [HC, Corollary 1] is a simple consequence of Theorem 9 or Corollary to Theorem 2.

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