

Foundations of the KKM Theory on Generalized Convex Spaces*

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Recently, we introduced a new concept of a generalized convex space. In this paper, from a coincidence theorem, we deduce far-reaching generalizations of the KKM theorem, the matching theorem, a whole intersection property, an analytic alternative, the Ky Fan minimax inequality, geometric or section properties, and others on generalized convex spaces. © 1997 Academic Press

1. INTRODUCTION

It is well known that the Brouwer fixed point theorem is equivalent to many results in nonlinear analysis and other fields such as the Sperner lemma [81], the Knaster–Kuratowski–Mazurkiewicz theorem [43], the Fan–Browder fixed point theorem [13, 25], Fan’s matching theorem [29, 30], minimax inequality [28], and many others. Most of those results are closely related to the so-called KKM maps. The *KKM theory* is the study of KKM maps and their applications. See Park [57, 58, 64]. As we have seen in the above, in the KKM theory, there exist mutually equivalent funda-

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mental theorems from which most of the other important results in the theory can be deduced.

In our previous work [68], we obtained a coincidence theorem for a class \mathfrak{A}_c^κ of admissible maps defined on generalized convex spaces and its applications to an abstract variational inequality, a new KKM type theorem, and fixed point theorems. The present paper is a continuation of [68].

In this paper, we give such fundamental theorems in the KKM theory related to the class \mathfrak{A}_c^κ of maps defined on generalized convex spaces which are adequate to establish theories on fixed points, coincidence points, KKM maps, variational inequalities, best approximations, and many others. Actually, from the basic coincidence theorem in [68] we deduce far-reaching generalizations of the KKM theorem, the matching theorem, a whole intersection property, an analytic alternative, the Ky Fan minimax inequality, geometric properties, and others for the maps in \mathfrak{A}_c^κ defined on generalized convex spaces.

Our new results extend, improve, and unify main theorems in many published works.

2. PRELIMINARIES

A *multifunction* (or *map*) $F: X \multimap Y$ is a function from a set X into the power set 2^Y of Y ; that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-}y = \{x \in X: y \in Fx\}$ for $y \in Y$. As usual, the set $\{(x, y): y \in Fx\}$ is called either the *graph* of F , or, simply, F . For $A \subset X$, let $F(A) = \bigcup\{Fx: x \in A\}$. A map $F: X \multimap Y$ is *compact* provided $F(X)$ is contained in a compact subset of a topological space Y . For any $B \subset Y$, the *lower inverse* and *upper inverse* of B under F are defined by

$$F^{-}(B) = \{x \in X: Fx \cap B \neq \emptyset\} \quad \text{and} \quad F^{+}(B) = \{x \in X: Fx \subset B\},$$

resp. The (*lower*) *inverse* of $F: X \multimap Y$ is the map $F^{-}: Y \multimap X$ defined by $x \in F^{-}y$ if and only if $y \in Fx$. Given two maps $F: X \multimap Y$ and $G: Y \multimap Z$, the *composite* $GF: X \multimap Z$ is defined by $(GF)x = G(Fx)$ for $x \in X$.

For topological spaces X and Y , a map $F: X \multimap Y$ is *upper semicontinuous* (u.s.c.) if it has nonempty values and, for each closed set $B \subset Y$, $F^{-}(B)$ is closed in X .

Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact.

Let $\bar{}$ denote the closure.

An *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T: X \multimap Y$ is one such that, for each T and each compact subset K of X , there exists a map $\Gamma \in \mathfrak{A}_c^\kappa(K, Y)$

satisfying $\Gamma x \subset Tx$ for all $x \in K$, where \mathfrak{A}_c consists of finite composites of maps in \mathfrak{A} , and \mathfrak{A} is a class of maps satisfying the following properties:

- (i) \mathfrak{A} contains the class \mathbb{C} of (single-valued) continuous functions;
- (ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
- (iii) for any polytope P , each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of \mathfrak{A} are continuous functions \mathbb{C} , the Kakutani maps \mathbb{K} (with convex values and codomain is a convex space), the Aronszajn maps \mathbb{M} (with R_δ values), the acyclic maps \mathbb{V} (with acyclic values), the Powers map \mathbb{V}_c , the O'Neill maps \mathbb{N} (continuous with values of one or m acyclic components, where m is fixed), the approachable maps \mathbb{A} in topological vector spaces, admissible maps of Górniewicz, permissible maps of Dzedzej, and many others. Note that \mathbb{V}_c^+ due to Park, Singh, and Watson [69] and \mathbb{K}_c^+ due to Lassonde [49] are examples of \mathfrak{A}_c^k . For details, see [61, 64, 65, 67].

More recently, the approximable maps $\mathbb{A}^k(X, Y)$ are due to Ben-El-Mechaiekh and Idzik [10], where X and Y are subsets of topological vector spaces (t.v.s.). They noted that if X is a convex subset of a locally convex Hausdorff t.v.s., then any u.s.c. map $T: X \rightarrow X$ with compact values belongs to \mathbb{A}^k whenever the functional values are all (1) convex, (2) contractible, (3) decomposable, or (4) ∞ -proximally connected.

For a nonempty set D , let $\langle D \rangle$ denote the set of all nonempty finite subset of D and $|D|$ the cardinality of D . Let Δ_n denote the standard n -simplex; that is,

$$\Delta_n = \left\{ u \in \mathbf{R}^{n+1}; u = \sum_{i=1}^{n+1} \lambda_i(u) e_i, \lambda_i(u) \geq 0, \sum_{i=1}^{n+1} \lambda_i(u) = 1 \right\},$$

where e_i is the i th unit vector in \mathbf{R}^{n+1} .

Let X be a set (in a vector space) and D a nonempty subset of X . Then (X, D) is called a *convex space* if convex hulls of any nonempty finite subsets of D are contained in X and X has a topology that induces the Euclidean topology on such convex hulls. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [46].

Let X be a topological space. A *c-structure* on X is given by a map $F: \langle X \rangle \rightarrow X$ such that

- (1) for all $A \in \langle X \rangle$, $F(A)$ is nonempty and contractible; and
- (2) for all $A, B \in \langle X \rangle$, $A \subset B$ implies $F(A) \subset F(B)$.

A pair (X, F) is then called a *c-space* by Horvath [38] and an *H-space* by Bardaro and Ceppitelli [4–6].

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X , a nonempty subset D of X , and a map $\Gamma: \langle D \rangle \rightarrow X$ with nonempty values such that

- (1) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and
- (2) for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A: \Delta_n \rightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$. For details on *G-convex spaces*, see [67, 68].

We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D \rangle$. For an $(X, D; \Gamma)$, a subset C of X is said to be *G-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

Note that Γ_A does not need to contain A for $A \in \langle D \rangle$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$.

Any convex space (X, D) becomes a *G-convex space* $(X, D; \Gamma)$ by putting $\Gamma_A = \text{co } A$, where co denotes the convex hull. An *H-space* (X, F) is a *G-convex space* $(X; \Gamma)$. In fact, by putting $\Gamma_A = F(A)$ for each $A \in \langle X \rangle$ with $|A| = n + 1$, there exists a continuous map $\phi_A: \Delta_n \rightarrow X$ such that for all $J \subset A$, $\phi_A(\Delta_J) \subset F(J)$ by Horvath [38, Theorem 1].

The other major examples of *G-convex spaces* are convex subsets of a t.v.s., metric spaces with Michael's convex structure [52], *S-contractible spaces* [70–72], Horvath's pseudo-convex spaces [36], Komiya's convex spaces [45], Bielawski's simplicial convexities [11], and Joó's pseudoconvex spaces [40].

For a convex space (X, D) , a map $G: D \rightarrow X$ is called a *KKM map* if $\text{co } N \subset G(N)$ for each $N \in \langle D \rangle$.

For a *G-convex space* $(X, D; \Gamma)$, a map $G: D \rightarrow X$ is called a *G-KKM map* if $\Gamma_N \subset G(N)$ for each $N \in \langle D \rangle$.

3. COINCIDENCE AND MATCHING THEOREMS

We begin with the following coincidence theorem [68, Theorem 1]:

THEOREM 1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S: D \rightarrow Y$, $T: X \rightarrow Y$ maps, and $F \in \mathfrak{A}_c^k(X, Y)$. Suppose that*

- (1.1) for each $x \in D$, Sx is compactly open in Y ;
- (1.2) for each $y \in F(X)$, $M \in \langle S^{-1}y \rangle$ implies $\Gamma_M \subset T^{-1}y$;
- (1.3) there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and

(1.4) *either*

(i) $Y \setminus K \subset S(M)$ for some $M \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that $F(L_N) \setminus K \subset S(L_N \cap D)$.

Then there exists an $\bar{x} \in X$ such that $F\bar{x} \cap T\bar{x} \neq \emptyset$.

From Theorem 1, we obtain the following Ky Fan type matching theorem for open covers:

THEOREM 2. Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, $S: D \rightarrow Y$, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that

(2.1) for each $x \in D$, Sx is compactly open;

(2.2) there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and

(2.3) *either*

(i) $Y \setminus K \subset S(M)$ for some $M \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that $F(L_N) \setminus K \subset S(L_N \cap D)$.

Then there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap \{Sx: x \in M\} \neq \emptyset$.

Proof. Let $H: Y \rightarrow X$ and $T: X \rightarrow Y$ be defined by $Hy = \bigcup \{\Gamma_M: M \in \langle S^{-1}y \rangle\}$ for $y \in Y$ and $Tx = H^{-1}x$ for $x \in X$. Then all of the requirements of Theorem 1 are satisfied, hence T and F have a coincidence point $x_0 \in X$; that is, $Tx_0 \cap Fx_0 \neq \emptyset$. For $y \in Tx_0 \cap Fx_0$, we have $x_0 \in T^{-1}y = \bigcup \{\Gamma_M: M \in \langle S^{-1}y \rangle\}$, and hence there exists a finite set $M \subset S^{-1}y \subset D$ such that $x_0 \in \Gamma_M$. Since $M \in \langle S^{-1}y \rangle$ implies $y \in Sx$ for all $x \in M$, we have $y \in Fx_0 \cap \bigcap \{Sx: x \in M\}$. This completes our proof.

Particular Forms. (1) The origin of Theorem 2 goes back to Fan [29, 30] for a convex subset $X = Y = D$ of a t.v.s. and $F = 1_X$.

(2) For a convex space X , Theorem 2 reduces to Park [64, Theorem 6] and, for \forall instead of \mathfrak{A}_c^κ to Park [57, Theorem 5], which includes his earlier results in [54, 55].

(3) For an H -space X and $F = 1_X$, Theorem 2 extends Park [59, Theorem 5; 60, Theorem 5] and Ding and Tan [23, Theorems 3, 6, 7, and Lemma 4].

From Theorem 2, we deduce a KKM type result of fundamental importance.

Recall that a family of sets is said to have the *finite intersection property* if the intersection of each finite subfamily is not empty.

COROLLARY. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $H: D \rightarrow Y$ such that, for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset H(N)$. Then the family $\{\overline{Hx}: x \in D\}$ has the finite intersection property.*

Proof. Suppose that $\bigcap \{\overline{Hx}: x \in M\} = \emptyset$ for some $M \in \langle D \rangle$. Put $Sx = Y \setminus \overline{Hx}$ for each $x \in D$. Then Sx satisfies (2.1), (2.2), and (2.3)(i), since $S(M) = Y$. By the conclusion of Theorem 2, $F(\Gamma_M) \cap \bigcap \{Sx: x \in M\} \neq \emptyset$; that is, $F(\Gamma_M) \not\subset \bigcup \{\overline{Hx}: x \in M\}$, hence

$$F(\Gamma_M) \not\subset H(M),$$

a contradiction. This completes our proof.

Particular Forms. (1) The origin of the Corollary is due to Fan [25] for a t.v.s. $E = X = Y$ and $F = 1_E$, and to Dugundji and Granas [24].

(2) For \mathbb{K}_c instead of \mathfrak{A}_c^κ , a convex set $X = D$ in a t.v.s., and a convex space Y , the Corollary reduces to Lassonde [47, Theorem 2].

(3) For a convex space X , the Corollary reduces to Park [64, Corollary 2].

4. THE KKM THEOREMS RELATED TO ADMISSIBLE MAPS

Theorem 2 can be stated in its contrapositive form and in terms of the complement Gx of Sx in Y . Then we obtain the following KKM theorem for G -convex spaces:

THEOREM 3. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G: D \rightarrow Y$ be a map such that*

(3.1) *for each $x \in D$, Gx is compactly closed in Y ;*

(3.2) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*

(3.3) *there exists a nonempty compact subset K of Y such that either*

(i) $\bigcap \{Gx: x \in M\} \subset K$ *for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that $F(L_N) \cap \bigcap \{Gx: x \in L_N \cap D\} \subset K$.*

Then $\overline{F(\overline{X})} \cap K \cap \bigcap \{Gx: x \in D\} \neq \emptyset$.

Proof. Suppose the conclusion does not hold. Then $\overline{F(\overline{X})} \cap K \subset S(D)$ where $Sx = Y \setminus Gx$ for $x \in D$. Note that (3.1) and (3.3) imply (2.1) and (2.3), resp. Therefore, by Theorem 2, there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap \{Sx: x \in M\} \neq \emptyset$; that is, $F(\Gamma_M) \not\subset G(M)$. This contradicts (3.2).

Remark. Condition (3.2) is equivalent to $\Gamma_N \subset F^+G(N)$; that is, the map F^+G is a G -KKM map. A KKM type theorem for this case different from Theorem 3 can be found in [57, Theorem 4].

Particular Forms. (1) The origin of Theorem 3 goes back to Sperner [81] and Knaster, Kuratowski, and Mazurkiewicz [43] for $X = Y = K = \Delta_n$ an n -simplex, D its set of vertices, and $F = 1_X$.

(2) For a convex space X , Theorem 3 reduces to Park [64, Theorem 7]. As Park noted in [57], a particular form [57, Theorem 3] of [64, Theorem 7] for \mathbb{V} instead of \mathfrak{A}_c^k includes earlier works of Fan [25, 29, 30], Lassonde [46], Chang [16], and Park [54, 55]. Moreover, in [63], Park showed that [57, Theorem 3] also extends a number of KKM type theorems due to Sehgal, Singh, and Whitfield [73], Lassonde [48], Shioji [80], Liu [51], Chang and Zhang [17], and Guillerme [33].

(3) For an H -space X and $F = 1_X$, Theorem 3 generalizes Horvath [36, Théorème 3.1 and Corollaire 3; 37, I, Theorem 1 and Corollary 1], Bardaro and Ceppitelli [4, Theorem 1], Ding and Tan [23, Corollary 1 and Theorem 8], Ding, Kim, and Tan [22, Lemma 1], Park [59, Theorems 1 and 4; 60, Theorems 1 and 3], and Ding [21, Theorems 3.1–3.4].

From Theorem 3, we have another whole intersection property as follows:

THEOREM 4. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^k(X, Y)$. Let $G: D \rightarrow Y$ be a map satisfying (3.1) and (3.3). Suppose that there exists a map $H: X \rightarrow Y$ satisfying*

$$(4.1) \text{ for each } x \in X, Fx \subset Hx; \text{ and}$$

$$(4.2) \text{ for each } y \in F(X), M \in \langle D \setminus G^{-1}y \rangle \text{ implies } \Gamma_M \subset X \setminus H^{-1}y.$$

Then $\overline{F(X)} \cap K \cap \bigcap \{Gx : x \in D\} \neq \emptyset$.

Proof. It suffices to show that (4.1) and (4.2) imply (3.2). Suppose that there exists an $N \in \langle D \rangle$ such that $F(\Gamma_N) \not\subset G(N)$; that is, there exist an $x \in \Gamma_N$ and a $y \in Fx$ such that $y \notin Gz$ for all $z \in N$. In other words, $N \in \langle D \setminus G^{-1}y \rangle$. By (4.2), $\Gamma_N \subset X \setminus H^{-1}y$. Since $x \in \Gamma_N$, we have $x \notin H^{-1}y$ or $y \notin Hx$. Since $y \in Fx$, this contradicts (4.1). This completes our proof.

Remark. Note that (3.2) and (4.2) are equivalent if $F = H$. We only need to show that (3.2) implies (4.2). Suppose that there exists a $y_0 \in F(X)$ and an $x_0 \in \Gamma_M \cap F^{-1}y_0$ for some $M \in \langle D \setminus G^{-1}y_0 \rangle$; that is, $y_0 \in Fx_0 \subset F(\Gamma_M)$, but $y_0 \notin G(M)$, which contradicts (3.2).

Here, in order to show that Theorems 1–4 are equivalent, the following suffices:

Proof of Theorem 1 using Theorem 4. Suppose that $Fx \cap Tx = \emptyset$ for all $x \in X$. Let $Gx = Y \setminus Sx$ for $x \in D$ and $Hx = Y \setminus Tx$ for $x \in X$. Then all

of the requirements of Theorem 4 are satisfied. Therefore, there exists a $y_0 \in \overline{F(X)} \cap K \cap \cap \{Gx: x \in D\}$; that is, $y_0 \in \overline{F(X)} \cap K$ such that $y_0 \notin Sx$ for all $x \in D$. This contradicts (1.3) in Theorem 1. This completes our proof.

Particular Forms. (1) The first particular form of Theorem 4 appears in Tarafdar [85] for a convex set $X = D = Y$ of a t.v.s. and $F = 1_X$.

(2) For a convex space X , Theorem 4 reduces to Park [64, Theorem 8].

(3) For an H -space X and $F = 1_X$, Theorem 4 extends Horvath [37, I, Theorem 2], Bardaro and Ceppitelli [4, Theorem 2], Ding and Tan [23, Theorem 5], and Park [59, Theorem 7; 60, Theorem 3].

5. ANALYTIC ALTERNATIVES

There are many equivalent and useful formulations of Theorems 1–4 in the KKM theory. In this section, we give analytic alternatives.

We begin, in this section, with the following useful reformulation of Theorem 1:

THEOREM 5. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^k(X, Y)$, $A, B \subset Z$ sets, $f: X \times Y \rightarrow Z$, $g: D \times Y \rightarrow Z$ functions, and K a nonempty compact subset of Y . Suppose that*

(5.1) *for each $x \in D$, $\{y \in Y: g(x, y) \in A\}$ is compactly open;*

(5.2) *for each $y \in F(X)$ and $M \in \langle \{x \in D: g(x, y) \in A\} \rangle$, we have $\Gamma_M \subset \{x \in X: f(x, y) \in B\}$; and*

(5.3) *either*

(i) *there exists an $M \in \langle D \rangle$ such that for each $y \in Y \setminus K$, $g(x, y) \in A$ for some $x \in M$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that for each $y \in F(L_N) \setminus K$, there exists an $x \in L_N \cap D$ satisfying $g(x, y) \in A$.*

Then either

(a) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $g(x, \hat{y}) \notin A$ for all $x \in D$;*
or

(b) *there exists an $(\hat{x}, \hat{y}) \in F$ such that $f(\hat{x}, \hat{y}) \in B$.*

Proof of Theorem 5 using Theorem 1. Consider the maps $S: D \rightarrow Y$ and $T: X \rightarrow Y$ given by

$$Sx = \{y \in Y: g(x, y) \in A\} \quad \text{for } x \in D$$

and

$$Tx = \{y \in Y: f(x, y) \in B\} \quad \text{for } x \in X.$$

Then (5.1), (5.2), and (5.3) imply (1.1), (1.2), and (1.4) in Theorem 1, resp. Suppose that (a) does not hold. Then, for each $y \in \overline{F(X)} \cap K$ there exists an $x \in D$ such that $g(x, y) \in A$; that is, $\overline{F(X)} \cap K \subset S(D)$. Hence (1.3) in Theorem 1 holds. Therefore, by Theorem 1, F and T have a coincidence point; that is, (b) holds.

Particular Forms. (1) The first form of Theorem 5 is due to Lassonde [46, Theorem 1.1'] for a convex space. Note that if F is single-valued, then Y is not necessarily Hausdorff. Lassonde used his result to generalize earlier works of Iohvidov [39], Fan [26], and Browder [13]. Applications of this kind of results to the Tychonoff fixed point theorem and the study of invariant subspaces of certain linear operators are given in [26, 39].

(2) For a convex space X , Theorem 5 reduces to Park [64, Theorem 9]. In particular, for $X = D$ and \forall instead of \mathfrak{A}_c^κ , Theorem 5 reduces to Park [57, Theorem 5].

(3) For an H -space X and $F = 1_X$, Theorem 5 extends Bardaro and Ceppitelli [4, Theorem 3], Tarafdar [86, Theorem 4], and Park [59, Theorem 8]. Note that Bardaro and Ceppitelli adopted a Riesz space instead of any set Z . For an H -space, Theorem 5 also extends Ding [21, Theorem 4.4].

From Theorem 5, we have the following analytic alternative, which is a basis of various minimax inequalities:

THEOREM 6. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, $\alpha, \beta \in \mathbf{R}$, $f: X \times Y \rightarrow \overline{\mathbf{R}}$, $g: D \times Y \rightarrow \overline{\mathbf{R}}$ extended real-valued functions, and K a nonempty compact subset of Y . Suppose that*

(6.1) *for each $x \in D$, $\{y \in Y: g(x, y) > \alpha\}$ is compactly open;*

(6.2) *for each $y \in F(X)$ and $M \in \langle \{x \in D: g(x, y) > \alpha\} \rangle$, we have $\Gamma_M \subset \{x \in X: f(x, y) > \beta\}$; and*

(6.3) *either*

(i) *there exists an $M \in \langle D \rangle$ such that for each $y \in Y \setminus K$, $g(x, y) > \alpha$ for some $x \in M$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that, for each $y \in F(L_N) \setminus K$, there exists an $x \in L_N \cap D$ satisfying $g(x, y) > \alpha$.*

Then either

(a) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $g(x, \hat{y}) \leq \alpha$ for all $x \in D$;*
or

(b) *there exists an $(\hat{x}, \hat{y}) \in F$ such that $f(\hat{x}, \hat{y}) > \beta$.*

Proof. Put $Z = \overline{\mathbf{R}}$, $A = (\alpha, \infty]$, and $B = (\beta, \infty]$ in Theorem 5.

Particular Forms. (1) The first form of Theorem 6 appeared in Ben-El-Mechaiekh *et al.* [8, 9] for a convex space $X = D = Y = K$ and $F = 1_X$. The authors used their result to deduce variational inequalities of Hartman and Stampacchia and Browder, a generalization of the Ky Fan minimax inequality, and others.

(2) For a convex space X , Theorem 6 reduces to Park [64, Theorem 10], which includes earlier results of Fan [28], Brézis, Nirenberg, and Stampacchia [12], Allen [1], Granas and Liu [31, 32], Tan [84], Lin [50], Ko and Tan [44], Deguire and Granas [20], Takahashi [83], Shih and Tan [79], Ding and Tan [23], Ben-El-Mechaiekh [7], Deguire [19], and Park [57].

(3) For an H -space X and $F = 1_X$, Theorem 6 extends Horvath [36, Theorem 5.1; 38, Proposition 5.1], Ding and Tan [23, Theorem 15], and Park [59, Theorem 9]. If we adopt a Riesz space instead of $\overline{\mathbf{R}}$ in Theorem 6, then we can obtain a generalized form of Bardaro and Ceppitelli [4, Theorem 4].

6. MINIMAX INEQUALITIES

From Theorem 6, we clearly have the following generalization of the Ky Fan minimax inequality:

THEOREM 7. *Under the hypothesis of Theorem 6, if $\alpha = \beta = \sup\{f(x, y) : (x, y) \in F\}$, then*

(c) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that*

$$g(x, \hat{y}) \leq \sup_{(x, y) \in F} f(x, y) \quad \text{for all } x \in D;$$

(d) *we have the minimax inequality*

$$\min_{y \in K} \sup_{x \in D} g(x, y) \leq \sup_{(x, y) \in F} f(x, y).$$

In order to show that Theorem 7 is equivalent to any of Theorems 1–6, we give the following:

Proof of Theorem 4 using Theorem 7. Define functions $g: D \times Y \rightarrow \mathbf{R}$ and $f: X \times Y \rightarrow \mathbf{R}$ by

$$g(x, y) = \begin{cases} 0 & \text{if } y \in Gx \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in D \times Y$ and

$$f(x, y) = \begin{cases} 0 & \text{if } y \in Hx \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in X \times Y$. Put $\alpha = \beta = 0$. Then (3.1) and (3.3) imply (6.1) and (6.3), resp. Further, (4.2) implies (6.2). In fact, $M \in \langle \{x \in X: g(x, y) > 0\} \rangle = \langle \{x \in X: g(x, y) = 1\} \rangle = \langle X \setminus G^{-}y \rangle$ implies $\Gamma_M \subset X \setminus H^{-}y = \{x \in X: f(x, y) = 1\} = \{x \in X: f(x, y) > 0\}$. Therefore, by Theorem 7, there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that

$$g(x, \hat{y}) \leq \sup_{(x, y) \in F} f(x, y) \quad \text{for all } x \in D.$$

However, $\sup\{f(x, y): (x, y) \in F\} \leq \sup\{f(x, y): (x, y) \in H\} = 0$ by (4.1) and the definition of f . Hence $g(x, \hat{y}) = 0$ for all $x \in D$; that is, $\hat{y} \in Gx$ for all $x \in D$. Therefore,

$$\overline{F(X)} \cap K \cap \bigcap \{Gx: x \in D\} \neq \emptyset.$$

This completes our proof.

Remark. Conclusion (d) can be written as

$$\min_{y \in K} \sup_{x \in D} g(x, y) \leq \inf_{F \in \mathfrak{A}_c^\kappa(X, Y)} \sup_{(x, y) \in F} f(x, y).$$

Particular Forms. (1) Theorem 7 originates from the Ky Fan minimax inequality [28] for a convex space $X = D = Y = K$, $f = g$, and $F = 1_X$. Fan applied his inequality to fixed point theorems, sets with convex sections, and potential theory. Later, the inequality became an important tool in nonlinear functional analysis, game theory, economic theory, and other fields.

(2) For a convex space X , Theorem 7 reduces to Park [64, Theorem 11]. In particular, Park [57, Theorem 9] for a convex space $X = D$ and \mathbb{V} instead of \mathfrak{A}_c^κ includes earlier works of Fan [28, 30], Brézis, Nirenberg, and Stampacchia [12], Takahashi [82, 83], Yen [87], Aubin [2], Ben-El-Mechaiekh *et al.* [8, 9], Tan [84], Shih and Tan [75, 76], Aubin and Ekeland [3], Lassonde [46], Granas and Liu [31, 32], Lin [50], Ha [34, 35], and Park [53].

(3) For an H -space X and $F = 1_X$, if we adopt a Riesz space instead of \mathbf{R} in Theorem 7, then we can obtain a generalized form of Bardaro and Ceppitelli [4, Corollary 1]. Note that Theorem 7 extends Park [59, Theorem 9] and Ding [21, Theorem 4.5].

The KKM theorem and the whole intersection theorem can be reformulated to minimax inequalities.

The following minimax inequality is equivalent to Theorem 3:

THEOREM 8. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, K a nonempty compact subset of Y , and $F \in \mathfrak{A}_c^k(X, Y)$. Let $\phi: D \times Y \rightarrow \overline{\mathbf{R}}$ be an extended real-valued function and $\gamma \in \overline{\mathbf{R}}$ such that*

(8.1) *for each $x \in D$, $\{y \in Y: \phi(x, y) \leq \gamma\}$ is compactly closed;*

(8.2) *for each $N \in \langle D \rangle$ and $y \in F(\Gamma_N)$, $\min\{\phi(x, y): x \in N\} \leq \gamma$;*
and

(8.3) *either*

(i) *there exists an $M \in \langle D \rangle$ such that for each $y \in Y \setminus K$, $\phi(x, y) > \alpha$ for some $x \in M$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that, for each $y \in F(L_N) \setminus K$, there exists an $x \in L_N \cap D$ satisfying $\phi(x, y) > \gamma$.*

Then (a) there exists a $\hat{y} \in \overline{F(\overline{X})} \cap K$ such that

$$\phi(x, \hat{y}) \leq \gamma \quad \text{for all } x \in D;$$

and (b) if $\gamma = \sup\{\phi(x, y): (x, y) \in F\}$, then we have the minimax inequality

$$\min_{y \in K} \sup_{x \in D} \phi(x, y) \leq \sup_{(x, y) \in F} \phi(x, y).$$

Proof of Theorem 8 using Theorem 3. Let $Gx = \{y \in Y: \phi(x, y) \leq \gamma\}$ for $x \in D$. Then (8.1) and (8.3) imply (3.1) and (3.3) clearly. We show that (8.2) implies (3.2). Suppose that there exists an $N \in \langle D \rangle$ such that $F(\Gamma_N) \not\subset G(N)$. Choose a $y \in F(\Gamma_N)$ such that $y \notin G(N)$, whence $\phi(x, y) > \gamma$ for all $x \in N$. Then $\min_{x \in N} \phi(x, y) > \gamma$, which contradicts (8.2). Therefore, by Theorem 3, there exists a $\hat{y} \in \overline{F(\overline{X})} \cap K$ such that $\hat{y} \in Gx$ for all $x \in D$; that is, $\phi(x, \hat{y}) \leq \gamma$ for all $x \in D$. This completes the proof of (a). Note that (b) clearly follows from (a).

Proof of Theorem 3 using Theorem 8. Define $\phi: D \times Y \rightarrow \overline{\mathbf{R}}$ by

$$\phi(x, y) = \begin{cases} 0 & \text{if } y \in Gx \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in D \times Y$. Put $\gamma = 0$ in Theorem 8. Then (3.1) clearly implies (8.1). We show that (3.2) implies (8.2). In fact, suppose that there exist an $N \in \langle D \rangle$ and a $y \in F(\Gamma_N)$ such that $\min\{\phi(x, y): x \in N\} > 0$. Then $y \notin Gx$ for all $x \in N$; that is, $F(\Gamma_N) \not\subset G(N)$, which contradicts (3.2). Moreover, (3.3) implies (8.3). Therefore, all of the requirements of Theo-

rem 8 are satisfied. Hence, there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $\phi(x, \hat{y}) = 0$ for all $x \in D$; that is, $\hat{y} \in \bigcap \{Gx: x \in D\}$. This completes our proof.

Remark. In the proof of the equivalency of Theorems 2 and 7 we used that (8.2) is equivalent to

(8.2)' the map $x \mapsto Gx = \{y \in Y: \phi(x, y) \leq \gamma\}$ satisfies Condition (3.2).

For similar arguments, see Ding, Kim, and Tan [22] and Chang and Zhang [17].

Particular Forms. (1) The first particular forms of Theorem 8 are due to Zhou and Chen [88, Theorem 2.11 and Corollary 2.13] for a convex space $X = D = Y = K$ and $F = 1_X$. Those results are applied to obtain a variation of the Ky Fan inequality, a saddle point theorem, and a quasi-variational inequality.

(2) For a convex space X , Theorem 8 reduces to Park [64, Theorem 12].

(3) For an H -space X and $F = 1_X$, Theorem 8 generalizes Park [60, Theorem 6].

7. GEOMETRIC OR SECTION PROPERTIES

In 1961, Fan [25] gave a “geometric” lemma which is the geometric equivalence of his version of the KKM theorem. In many of his works in the KKM theory, Fan actually based his arguments mainly on the geometric or section property of a convex space. We now deduce two geometric forms of Theorem 4. The first one is as follows:

THEOREM 9. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^k(X, Y)$, $A \subset B \subset X \times Y$, and $C \subset D \times Y$. Suppose that*

(9.1) *for each $x \in D$, $\{y \in Y: (x, y) \in C\}$ is compactly closed in Y ;*

(9.2) *for each $y \in F(X)$ and $M \in \langle \{x \in D: (x, y) \notin C\} \rangle$, we have $\Gamma_M \subset \{x \in X: (x, y) \notin B\}$;*

(9.3) *A contains the graph of F ; and*

(9.4) *there exists a nonempty compact subset K of Y such that either*

(i) $\bigcap_{x \in M} \{y \in Y: (x, y) \in C\} \subset K$ *for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, a compact G -convex subset L_N of X containing N such that $F(L_N) \cap \bigcap_{x \in L_N \cap D} \{y \in Y: (x, y) \in C\} \subset K$.*

Then there exists a $y_0 \in \overline{F(X)} \cap K$ such that $D \times \{y_0\} \subset C$.

Proof of Theorem 9 using Theorem 4. For each $x \in D$, let

$$Gx = \{y \in Y: (x, y) \in C\},$$

which is compactly closed by (9.1). Moreover, for each $x \in X$, let $Hx = \{y \in Y: (x, y) \in B\}$. Then $A \subset B$, (9.2), and (9.3) imply (4.1) and (4.2). Since (9.4) clearly implies (3.3), G satisfies all of the requirements of Theorem 4. Therefore, we have

$$\overline{F(X)} \cap K \cap \bigcap \{Gx: x \in D\} \neq \emptyset.$$

Hence, there exists a $y_0 \in \overline{F(X)} \cap K$ such that $y_0 \in \bigcap \{Gx: x \in D\}$; that is, $D \times \{y_0\} \subset C$.

Particular Forms. (1) The original form of Theorem 9 due to Fan [25] is the case for a convex subset $X = D = Y$ of a t.v.s., $A = B = C$, and $F = 1_X$. Fan's geometric lemma has many applications, among which are fixed point theorems, theorems on minimax, existence of equilibrium points, extensions of monotone sets, a fundamental existence theorem in potential theory, variational inequalities, and many others.

(2) For a convex space X , Theorem 9 reduces to Park [64, Theorem 13], whose particular form [63, Theorem 12] for a convex space $X = D$ and $F \in \mathbb{V}(X, Y)$ extends earlier results of Fan [25, 27, 28]. Takahashi [82], Shih and Tan [76], Lin [50], Park [53, 57], Ha [34], Shioji [80], and Sehgal, Singh, and Whitfield [73].

(3) For an H -space X , Theorem 9 extends Chen [18, Theorem 1] and Ding [21, Theorem 4.1].

The following form of Theorem 9 is also widely used in the KKM theory:

THEOREM 10. *Let $(X, D; \Gamma)$ be a G -convex space, Y a Hausdorff space, K a nonempty compact subset of Y , $F \in \mathfrak{A}_c^k(X, Y)$, $B \subset A \subset X \times Y$, and $C \subset D \times Y$. Suppose that*

(10.1) *for each $x \in D$, $\{y \in Y: (x, y) \in C\}$ is compactly open in Y ;*

(10.2) *for each $y \in F(X)$ and $M \in \langle \{x \in X: (x, y) \in C\} \rangle$, we have $\Gamma_M \subset \{x \in X: (x, y) \in B\}$;*

(10.3) *for each $y \in \overline{F(X)} \cap K$, there exists an $x \in D$ such that $(x, y) \in C$; and*

(10.4) *either*

(i) $\bigcap_{x \in M} \{y \in Y: (x, y) \notin C\} \subset K$ *for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact G -convex subset L_N of X containing N such that $F(L_N) \cap \bigcap_{x \in L_N \cap D} \{y \in Y: (x, y) \notin C\} \subset K$.*

Then there exist an $x_0 \in X$ and a $y_0 \in Fx_0$ such that $(x_0, y_0) \in A$.

Proof of Theorem 10 using Theorem 9. Consider Theorem 9 with the complements (A^c, B^c, C^c) instead of (A, B, C) . Then (9.1), (9.2), and (9.4) are satisfied automatically. Since (10.3) is the negation of the conclusion of Theorem 9, we should have the negation of (9.3). Therefore, the conclusion follows.

Proof of Theorem 1 using Theorem 10. Let $A = B$ be the graph of T and C the graph of S . Then (1.1)–(1.4) in Theorem 1 imply (10.1)–(10.4). Therefore, by Theorem 10, there exists an $(x_0, y_0) \in F$ such that $(x_0, y_0) \in A$; that is, F and T have a coincidence point.

Consequently, Theorems 1–10 are all equivalent to each other.

Particular Forms. (1) The origin of Theorem 10 goes back to Fan [28] for a convex subset $X = D = Y = K$ of a t.v.s., $A = B = C$, and $F = 1_X$. This is equivalent to the Fan-Browder fixed point theorem [13–15].

(24) For a convex space X , Theorem 10 reduces to Park [64, Theorem 14] and, for a convex space $X = D$ and $F \in \mathbb{V}(X, Y)$, to [63, Theorem 13], which extends earlier works of Fan [29], Shih and Tan [76–79], and Park [53, 55].

8. OPEN-VALUED VERSION OF KKM THEOREM

In order to obtain the finite intersection property for an open-valued KKM map, we need the following motivated by Shih [74, Theorem 1] and Park and Kim [66, Theorem 5]:

LEMMA. *Let $(X, D; \Gamma)$ be a Hausdorff G -convex space, $D \in \langle X \rangle$, Y a regular space, and $F: X \multimap Y$ a compact-valued u.s.c. map. If $G: D \multimap Y$ is an open-valued map such that*

- (1) *for each $J \in \langle D \rangle$, $F(\Gamma_J) \subset G(J)$,*

then there is a closed-valued map $H: D \multimap Y$ such that $Hx \subset Gx$ for all $x \in D$ and

- (2) *$F\phi_D(\Delta_J) \subset H(J)$ for each $J \subset D$ where Δ_J is the face of Δ_n corresponding to J and ϕ_D a continuous map such that $\phi_D(\Delta_J) \subset \Gamma_J$.*

Proof. For any $y \in G(D)$, let

$$H_y = \bigcap \{Gx : y \in Gx\};$$

then H_y is an open set in Y containing y . By the regularity of Y , there exists an open neighborhood U_y of y in Y such that

$$y \in U_y \subset \bar{U}_y \subset H_y.$$

Clearly, for any $J \subset D$, we have

$$G(J) \subset \cup \{U_y: y \in G(J)\}$$

and so by (1), $\{U_y: y \in G(J)\}$ is an open cover of $F\phi_D(\Delta_J)$ because $F\phi_D(\Delta_J) \subset F(\Gamma_J)$. Since $F\phi_D(\Delta_J)$ is compact, there exists a $B_J \in \langle G(J) \rangle$ such that

$$F\phi_D(\Delta_J) \subset \cup \{U_y: y \in B_J\}.$$

Let $B = \cup \{B_J: J \subset D\}$. Define $H: D \rightarrow Y$ by

$$Hx = \cup \{\bar{U}_y: y \in B \cap Gx\}$$

for each $x \in D$. Then Hx is closed in Y and $Hx \subset Gx$ for each $x \in D$, since $\bar{U}_y \subset H_y \subset Gx$ if $y \in Gx$. For each $J \subset D$ and $z \in F\phi_D(\Delta_J)$, we have $z \in U_y$ for some $y \in B_J \subset G(J) \cap B$; that is, $y \in Gx \cap B$ for some $x \in J$. Hence

$$F\phi_D(\Delta_J) \subset H(J).$$

This completes our proof.

The following is an open-valued version of the KKM theorem for G -convex spaces:

THEOREM 11. *Let $(X, D; \Gamma)$ be a G -convex space, Y a T_1 regular space, $F \in \mathfrak{A}_c^*(X, Y)$, and $G: D \rightarrow Y$ such that*

- (1) *for each $x \in D$, Gx is open; and*
- (2) *for any $N \in \langle D \rangle$, $\Gamma_N \subset F^+G(N)$.*

Then $\{Gx: x \in D\}$ has the finite intersection property.

Proof. Suppose that $\cap \{Gx: x \in M\} = \emptyset$ for some $M \in \langle D \rangle$. By Lemma, for each $M \in \langle D \rangle$, there exists a map $H: M \rightarrow X$ with closed values such that $Hx \subset (F^+G)x$ for each $x \in M$ and Condition (2) in the Lemma holds. And so $F(\Gamma_M) \cap \cap \{Hx: x \in M\} = \emptyset$. Then for $\tilde{F} \in \mathfrak{A}_c(\phi_M(\Delta_n), Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in \phi_M(\Delta_n)$, $\tilde{F}\phi_M(\Delta_n) \subset T(M) = Y$, where $|M| = n + 1$ and $Tx = Y \setminus Hx$ for each $x \in M$. Let $\{\lambda_i\}_{i=1}^{n+1}$ be the partition of unity subordinated to this cover $\{Tx_i\}_{i=1}^{n+1}$ of the compact subset $\tilde{F}\phi_M(\Delta_n)$ of Y . Define $p: \tilde{F}\phi_M(\Delta_n) \rightarrow \Delta_n$ by

$$py = \sum_{i=1}^{n+1} \lambda_i(y) e_i = \sum_{i \in N_y} \lambda_i(y) e_i$$

for $y \in \tilde{F}\phi_M(\Delta_n)$ where $i \in N_y \Leftrightarrow \lambda_i(y) \neq 0 \Rightarrow y \in Tx_i$.

Then $p\tilde{F}\phi_M \in \mathfrak{A}_c(\Delta_n, \Delta_n)$ has a fixed point $z_0 \in \Delta_n$; that is, $z_0 \in (p\tilde{F}\phi_M)z_0$. So there is a $y_0 \in (\tilde{F}\phi_M)z_0$ such that $py_0 = z_0$ and $y_0 \in \tilde{F}x_0$ where $x_0 = \phi_M z_0$. If $i \in N_{y_0}$, then $y_0 \in Tx_i$, and

$$y_0 \in \tilde{F}x_0 \cap \bigcap \{Tx_i : i \in N_{y_0}\} \neq \emptyset.$$

So $\tilde{F}\phi_M(\Delta_{N_{y_0}}) \cap \bigcap \{Tx : x \in N_{y_0}\} \neq \emptyset$, and hence $F\phi_M(\Delta_{N_{y_0}}) \not\subset H(N_{y_0})$, where $\Delta_{N_{y_0}}$ is the face of Δ_n corresponding to N_{y_0} . This contradiction completes our proof.

Remark. If F is a continuous single-valued map, then the T_1 regularity of Y is not necessary. Compare with Park [59, Theorem 14].

Particular Forms. (1) The origin of Theorem 11 is due to Kim [41] for a convex subset $X = Y$ of a t.v.s. and $F = 1_X$.

(2) For a convex space X , Theorem 11 reduces to Park [64, Corollary 3], which includes earlier results of Kim [41, 42], Park [63, Theorem 10], and Lassonde [48, Theorem 1].

(3) for H -spaces, Theorem 11 reduces to Park and Kim [66, Theorem 6].

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