BEST APPROXIMATIONS AND FIXED POINTS OF MULTIFUNCTIONS WHOSE DOMAINS AND RANGES HAVE DIFFERENT TOPOLOGIES

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1. Introduction

The use of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz [KKM] to the fixed point theory of upper semicontinuous multifunctions with convex values is essential. See [L]. Recently this theory has been developed very extensively in three major directions.

In the second author's works [P5,6,10, PB2], very general fixed point theorems for convex-valued generalized upper-hemicontinuous multifunctions defined on a convex subset of a topological vector space E were obtained by using an existence theorem of maximizable u.s.c. convex functions on a convex space. Such method was initiated by Fan [F3], Simons [S], Bellenger [Be], and Park and Bae [PB1].

Moreover, in [P1,2,4], it was shown that certain fixed point theorems can be deduced from variational inequalities. This method of proving fixed point theorems was originated from Browder [B1] and Halpern [Hl].

Further, another way of proving fixed point theorems for convex-valued multifunctions is the best approximation method, which can be seen in the works of Fan [F1,2], Reich [R1-3], Sehgal and Singh [SS], Park [P3], Ding and Tan [DT1,2],

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and many others. However, this method seems to be a little ineffective. For example, recent works of Ding and Tan [DT2,3] could give results only for continuous multifunctions.

Note that the first and second methods heavily depend on the topological dual E^* of E. On the other hand, the best approximation method depends on (weakly) continuous seminorms. However, those all three methods are based on certain equivalent formultaitons of the KKM theorem. One of such forms, for example, a minimax inequality, is used to deduce fixed point theorems for large classes of multifunctions including the class of composites of acyclic maps in [P6-9].

In the present paper, we discuss mainly the best approximation method for convex-valued multifunctions. Such method was initiated by Fan [F1] and followed by [F2, DT1-3, R1-3, SS, SSS, SSW, P3]. On the other hand, Browder [B2] obtained some sharpened forms of the Schauder fixed point theorem based on variations of best approximation theorems. Browder's work was further extended by several authors [DT1, R3, ST, T].

Recently, Ding and Tan [DT1] gave generalizations of the Browder type results due to Browder [B2], Jiang [J], and Shih and Tan [ST] for inward or outward multifunctions defined on non-compact convex sets. Moreover, Ding and Tan [DT2] obtained some Ky Fan type best approximation theorems and several fixed point theorems for continuous multifuncitons defined on non-compact convex subsets of a topological vector space having sufficiently many linear functionals. They also added same type of theorems for upper semicontinuous multifunctions defined on weakly compact convex subsets of a locally convex Hausdorff topological vector space. One of the main features of such research is that the domain and range of the concerned multifunction may have different topologies. Such idea was first given by Kapoor [K] and later followed by Sehgal et al. [SSS], Roux and Singh [RS], and Park [P6, PB2].

Now, we are mainly concerned with best approximation and fixed point theorems on multifunctions whose domains and ranges have different topologies. We show that our general best approximation theorems imply fixed point theorems for continuous or upper semicontinuous multifunctions.

In Section 3, we establish some generalized best approximation theorems, which can be applied to fixed point theorems for inward or outward multifunctions defined on convex subsets of topological vector spaces not necessarily Hausdorff. Consequently, the Browder type sharpened forms [B2] due to Ding and Tan [DT1, Section 6] are extended in several aspects.

Section 4 deals with set-valued generalizations of Fan's best approximation theorem for continuous multifunctions and its application to fixed point theorems. Thus, the main results of Ding and Tan [DT2, Section 3] are generalized under much more weaker assumptions.

In Section 5, we obtain another type of best approximation theorems for upper semicontinuous multifunctions with compact values. This is applied to noncompact versions of results of Ding and Tan [DT2, Section 4] for not necessarily locally convex topological vector spaces.

2. Preliminaries

Let X be a nonempty set, 2^X the family of its nonempty subsets, and $\langle X \rangle$ the family of its nonempty finite subsets. A function $F : X \to 2^Y$ is said to be a multifunction or a map.

For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be

(i) *upper semicontinuous* (u.s.c.) if for each $x \in X$ and each neighborhood V of Fx in Y, there exists a neighborhood U of x in X such that $F(U) \subset V$;

(ii) *lower semicontinuous* (l.s.c.) if for each $x \in X$ and each open set V of Y such that $Fx \cap V \neq \emptyset$, there exists a neighborhood U of x in X such that $Fz \cap V \neq \emptyset$ for each $z \in U$; and

(iii) *continuous* if it is u.s.c. and l.s.c.

For a (real or complex) topological vector space E, let

E^* be its topological dual,

$k(E)$ the family of nonempty compact subsets of E,

$kc(E)$ the subfamily of convex subsets in $k(E)$,

$w = w(E, E^*)$ the weak topology of E, and

$S(E, w)$ the family of weakly continuous seminorms on E.

For subsets of E, co, $\overline{}$, Bd, and Int denote the convex hull, closure, boundary, and interior, resp.

We say that E^* *separates points* of E whenever for each nonzero $x \in E$, there exists an $h \in E^*$ such that $h(x) \neq 0$. This implies that for each nonzero $x \in E$, there exists a $p \in S(E, w)$ such that $p(x) > 0$.

For an $X \in 2^E$ and $x \in E$, the *inward set* and *outward set* of X at x, denoted by $I_X(x)$ and $O_X(x)$, are defined by

$$I_X(x) = x + \bigcup_{r>0} r(X - x) \quad \text{and} \quad O_X(x) = x + \bigcup_{r<0} r(X - x),$$

resp. Their closures $\overline{I}_X(x)$ and $\overline{O}_X(x)$ are called the *weakly inward set* and *weakly outward set*, resp. If $x \in \operatorname{Int} X$, then x is an internal point and $E = I_X(x) = O_X(x)$.

A multifunction $F : X \to 2^E$ is said to be *inward* and *outward*, resp., if $Fx \cap I_X(x) \neq \emptyset$ and $Fx \cap O_X(x) \neq \emptyset$, resp., for each $x \in \operatorname{Bd} X \backslash Fx$. Similarly, *weakly inward* and *weakly outward* multifunction can be defined.

A *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of X is said to be *c-compact* if for each $N \in \langle X \rangle$, there is an $L_N \in kc(X)$ such that $L \cup N \subset L_N$, where $kc(X)$ is defined similarly as above. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$. See Lassonde [L].

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$ on a topological space X is *lower* [*upper*, resp.] *semicontinuous* (l.s.c.) [u.s.c., resp.] if $\{x \in X : fx > r\}$ [$\{x \in X : fx < r\}$, resp.] is open for each $r \in \mathbb{R}$.

The following two lemmas can be seen in [A]:

Lemma 1. *Let X and Y be topological spaces, $g : X \times Y \to \mathbb{R}$ l.s.c., and $F : X \to k(Y)$ u.s.c. Then the funciton $U : X \to [-\infty, \infty)$ defined by*

$$U(x) = \inf_{y \in Fx} g(x, y)$$

is l.s.c.

Lemma 2. *Let X and Y be topological spaces, $g : X \times Y \to \mathbb{R}$ u.s.c., and $F : X \to 2^Y$ l.s.c. Then the function $V : X \to [-\infty, \infty)$ defined by*

$$V(x) = \inf_{y \in Fx} g(x, y)$$

is u.s.c.

The following minimax inequality is an equivalent formulation of the generalized KKM theorem recently due to the second author [P7, Theorem 12]:

Theorem 0. *Let X be a convex space, Y a Hausdorff space, $T \in \mathfrak{A}_c^\kappa(X, Y)$, and $K \in k(Y)$. Suppose that a function $\phi : X \times Y \to \overline{\mathbb{R}}$ and a $\gamma \in \overline{\mathbb{R}}$ satisfy the following:*

(0.1) *for each $x \in X$, $\{y \in Y : \phi(x, y) \le \gamma\}$ is compactly closed;*

(0.2) *for each $N \in \langle X \rangle$ and $y \in T(\operatorname{co} N)$, $\min\{\phi(x, y) : x \in N\} \le \gamma$; and*

(0.3) *for each $N \in \langle X \rangle$, there exists an $L_N \in kc(X)$ containing N such that, for each $y \in T(L_N) \backslash K$, there exists an $x \in L_N$ satisfying $\phi(x, y) > \gamma$.*

Then there exists a $\hat{y} \in \overline{T(X)} \cap K$ such that

$$\phi(x, \hat{y}) \le \gamma \quad \text{for all} \quad x \in X.$$

Here, $\mathfrak{A}_c^\kappa$ represents very large classes of maps including the class of u.s.c. maps $F : X \to 2^Y$ with compact acyclic values; and if T is single-valued the Hausdorffness of Y is not necessary.

In this paper, we mainly use the particular form of Theorem 0 for $X = Y$, $T = 1_X$, and $\gamma = 0$. Even in this case, Theorem 0 improves Ding and Tan [DT1, Theorem 1; DT2, Lemma 8]. In [DT1, Section 4], some equivalent formulations of this case are given, but those are all consequences of results of Park [P7]. Some other particular or equivalent forms of Theorem 0 are mentioned in [P7].

3. The Browder type sharpened forms

In this section, we obtain very general forms of the Browder type sharpened forms of the Schauder fixed point theorem for inward or outward multifunctions defined on convex subsets of topological vector spaces not necessarily Hausdorff. The main feature of our new theorems is that the domain and range of a multifunction may have different topologies.

We begin with the following basis of best approximation or fixed point theorems for continuous multifuncitons:

Theorem 3.1. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that the inclusion $i_X : X \hookrightarrow E$ is continuous, and $F : X \to kc(E)$. Let $p : X \times E \to [0, \infty)$ be a function such that, for each $x \in X$, $p(x, \cdot)$ is a continuous convex function on E. Suppose that*

(1.1) *for each $y \in X$,*

$$\{x \in X : \inf_{u \in Fx} p(x, x - u) \leq \inf_{v \in Fx} p(x, y - v)\}$$

is compactly closed in X; and

(1.2) *for each $x \in X \backslash K$ with $\inf_{u \in Fx} p(x, x - u) > 0$, there exist a $z \in \overline{I}_L(x)$ and a $v \in Fx$ such that*

$$p(x, z - v) < \inf_{u \in Fx} p(x, x - u).$$

Then either

(a) *there exist an* $\hat{x} \in X \backslash K$ *and a* $\hat{u} \in F\hat{x}$ *such that* $p(\hat{x}, \hat{x} - \hat{u}) = 0$; *or*

(b) *there exist an* $\hat{x} \in K$ *and a* $\hat{u} \in F\hat{x}$ *such that*

$$p(\hat{x}, \hat{x} - \hat{u}) \leq p(\hat{x}, y - v)$$

for all $y \in \overline{I}_X(\hat{x})$ *and all* $v \in F\hat{x}$.

Proof. If $\inf_{u \in Fx} p(x, x - u) = 0$ for some $x \in X \backslash K$, then (a) holds since $p(x, \cdot)$ is continuous and $Fx \in k(E)$.

Suppose that $\inf_{u \in Fx} p(x, x - u) > 0$ for all $x \in X \backslash K$. Define a function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \inf_{u \in Fx} p(x, x - u) - \inf_{v \in Fx} p(x, y - v)$$

for $(x, y) \in X \times X$. We use Theorem 0 with $X = Y$, $T = 1_X$, $\gamma = 0$, and interchanging the roles of x and y.

(0.1) For each $y \in X$, $\{x \in X : \phi(x, y) \leq 0\}$ is compactly closed by (1.1).

(0.2) For each $N \in \langle X \rangle$ and $x \in \operatorname{co} N$, $\min\{\phi(x, y) : y \in N\} \leq 0$.

Suppose, on the contrary, that there exist an $N = \{y_1, y_2, \ldots, y_n\} \in \langle X \rangle$ and an $\overline{x} = \sum_{i=1}^n r_i y_i \in \operatorname{co} N$ with $\sum_{i=1}^n r_i = 1$ and $r_i > 0$ for all i such that

$$\phi(\overline{x}, y) = \inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) - \inf_{v \in F\overline{x}} p(\overline{x}, y - v) > 0$$

for all $y \in N$; that is,

$$\inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) > \inf_{v \in F\overline{x}} p(\overline{x}, y_i - v)$$

for all i, $1 \leq i \leq n$. Since $p(\overline{x}, \cdot)$ is continuous on E and $F\overline{x} \in k(E)$, for each $y_i \in N$, there exists a $v_i \in F\overline{x}$ such that

$$\inf_{v \in F\overline{x}} p(\overline{x}, y_i - v) = p(\overline{x}, y_i - v_i).$$

Let $\overline{v} = \sum_{i=1}^{n} r_i v_i \in F\overline{x}$ since $F\overline{x}$ is convex. Since $p(x, \cdot)$ is convex,

$$\begin{aligned}\inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) &\leq p(\overline{x}, \overline{x} - \overline{v}) = p(\overline{x}, \sum_{i=1}^{n} r_i(y_i - v_i)) \\ &\leq \sum_{i=1}^{n} r_i p(\overline{x}, y_i - v_i) = \sum_{i=1}^{n} r_i \inf_{v \in F\overline{x}} p(\overline{x}, y_i - v) \\ &< \left(\sum_{i=1}^{n} r_i\right) \inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) = \inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u),\end{aligned}$$

which is a contradiction.

(0.3) Since L is c-compact, for each $N \in \langle X \rangle$, there exists a set $L_N \in kc(X)$ containing $L \cup N$. Therefore, in order to get (0.3), it suffices to show that, for each $x \in X \backslash K$, there exists a $y \in [x, L]$ satisfying $\phi(x, y) > 0$.

Suppose that there exists an $\overline{x} \in X \backslash K$ such that $\phi(\overline{x}, z) \leq 0$ or, equivalently,

$$\inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) \leq \inf_{v \in F\overline{x}} p(\overline{x}, z - v)$$

for all $z \in [\overline{x}, L]$. Since $p(\overline{x}, \cdot)$ is continuous on E and $F\overline{x} \in k(E)$, there exists a $\overline{u} \in F\overline{x}$ such that

$$\inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) = p(\overline{x}, \overline{x} - \overline{u}).$$

Therefore,

$$(*) \qquad p(\overline{x}, \overline{x} - \overline{u}) \leq p(\overline{x}, z - v)$$

holds for all $z \in [\overline{x}, L]$ and all $v \in F\overline{x}$. Since $\overline{x} \in X \backslash K$ and $p(\overline{x}, \overline{x} - \overline{u}) > 0$, by (1.2), there exist a $z_1 \in \overline{I}_L(\overline{x})$ and a $v_1 \in F\overline{x}$ such that

$$p(\overline{x}, z_1 - v_1) < p(\overline{x}, \overline{x} - \overline{u}).$$

Therefore, $z_1 \notin [\overline{x}, L]$. Since $p(\overline{x}, \cdot)$ is continuous on E, we may assume that $z_1 \in I_L(\overline{x}) \backslash [\overline{x}, L]$. Hence, $z_1 = \overline{x} + \lambda(z - \overline{x})$ for some $z \in L$ and $\lambda > 1$ as $[\overline{x}, L]$ is convex. Note that

$$z = \frac{1}{\lambda} z_1 + (1 - \frac{1}{\lambda})\overline{x} \in L.$$

Since $F\overline{x}$ is convex and $v_1, \overline{u} \in F\overline{x}$, we have

$$w = \frac{1}{\lambda} v_1 + (1 - \frac{1}{\lambda})\overline{u} \in F\overline{x}.$$

Therefore,

$$\begin{aligned} p(\overline{x}, z - w) &= p(\overline{x}, \frac{1}{\lambda}(z_1 - v_1) + (1 - \frac{1}{\lambda})(\overline{x} - \overline{u})) \\ &\leq \frac{1}{\lambda} p(\overline{x}, z_1 - v_1) + (1 - \frac{1}{\lambda}) p(\overline{x}, \overline{x} - \overline{u}) \\ &< \frac{1}{\lambda} p(\overline{x}, \overline{x} - \overline{u}) + (1 - \frac{1}{\lambda}) p(\overline{x}, \overline{x} - \overline{u}) \\ &= p(\overline{x}, \overline{x} - \overline{u}), \end{aligned}$$

which contradicts $(*)$.

Now ϕ satisfies all of the requirements of Theorem 0. Therefore, there exists an $\hat{x} \in K$ such that $\phi(\hat{x}, y) \leq 0$; that is,

$$\inf_{u \in F\hat{x}} p(\hat{x}, \hat{x} - u) \leq p(\hat{x}, y - v)$$

for all $y \in X$ and all $v \in F\hat{x}$. So, as above, there exists a $\hat{u} \in F\hat{x}$ such that

$$p(\hat{x}, \hat{x} - \hat{u}) \leq p(\hat{x}, y - v)$$

for all $y \in \overline{I}_X(\hat{x})$ and all $v \in F\hat{x}$. This completes our proof.

Remarks. 1. Instead of F and p, by consisering $G : X \to kc(E)$ defined by $Gx = 2x - Fx$ for $x \in X$ and $q : X \times E \to \mathbb{R}$ defined by $q(x, y) = p(x, -y)$ for $(x, y) \in X \times E$, the inward sets in Theorem 3.1 can be replaced by the corresponding outward sets.

2. Note that, in Theorem 3.1, the topology of the convex space X can be different from the relative topology with respect to E.

From Lemmas 1 and 2, we obtain the following:

Lemma 3. *Let X be a convex space, E a t.v.s. containing X as a subset such that the inclusion $i_X : X \hookrightarrow E$ is continuous, $p : X \times E \to [-\infty, \infty)$ a function continuous on $C \times E$ for each $C \in k(X)$, and $F : X \to 2^E$.*

(i) *If F is l.s.c. on $C \in k(X)$, then, for each $y \in X$, $x \mapsto \inf_{v \in Fx} p(x, y - v)$ is u.s.c. on C.*

(ii) *If F is u.s.c. on $C \in k(X)$ and compact-valued, then $x \mapsto \inf_{u \in Fx} p(x, x-u)$ is l.s.c. on C.*

(iii) *If F is continuous on each $C \in k(X)$ and compact-valued, then*

$$\{x \in X : \inf_{u \in Fx} p(x, x - u) \leq \inf_{v \in Fx} p(x, y - v)\}$$

is compactly closed in X.

Proof. (i) For each $y \in X$, let $g(x, v) = p(x, y - v)$ for $(x, v) \in X \times E$. Then for each $C \in k(X)$, $g|_{C \times E}$ is u.s.c. and $F|_C : C \to 2^E$ is l.s.c. Therefore, by Lemma 2, $x \mapsto \inf_{v \in Fx} p(x, y - v)$ is u.s.c. on C.

(ii) Let $g(x, u) = p(x, x - u)$ for $(x, u) \in X \times E$. Since i_X is continuous and $p : C \times E \to \mathbb{R}$ is continuous on each $C \in k(X)$, $g|_{C \times E}$ is l.s.c. Since $F|_C : C \to k(E)$ is u.s.c., by Lemma 1, $x \mapsto \inf_{u \in Fx} p(x, x - u)$ is l.s.c. on C.

(iii) From (i) and (ii), for each $y \in X$,

$$x \mapsto \inf_{u \in Fx} p(x, x - u) - \inf_{v \in Fx} p(x, y - v)$$

is l.s.c. on each $C \in k(X)$.

From Theorem 3.1 and Lemma 3, we have the following:

Theorem 3.2. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that $i_X : X \hookrightarrow E$ is continuous, $p : X \times E \to [0, \infty)$, and $F : X \to kc(E)$ such that*

(i) *p is continuous on $C \times E$ and F is continuous on C for each $C \in k(X)$; and*

(ii) *for each $x \in X$, $p(x, \cdot)$ is a convex function on E.*

Suppose that the following holds:

(2.1) *for each $x \in K$ with $\inf_{u \in Fx} p(x, x-u) > 0$, there exist a $z \in \overline{I}_X(x)$ and a $v \in Fx$ such that*

$$p(x, z-v) < \inf_{u \in Fx} p(x, x-u);$$

and

(2.2) *for each $x \in X \backslash K$ with $\inf_{u \in Fx} p(x, x-u) > 0$, there exist a $z \in \overline{I}_L(x)$ and a $v \in Fx$ such that*

$$p(x, z-v) < \inf_{u \in Fx} p(x, x-u).$$

Then there exist an $\hat{x} \in X$ and a $\hat{u} \in F\hat{x}$ such that $p(\hat{x}, \hat{x} - \hat{u}) = 0$.

Proof. We use Theorem 3.1. By the continuity of p and F, (1.1) follows from Lemma 3. Moreover, (2.2) is the same as (1.2). Therefore, the conclusion (a) or (b) of Theorem 3.1 holds. Note that the $\hat{x} \in K$ and $\hat{u} \in F\hat{x}$ also satisfy $p(\hat{x}, \hat{x} - \hat{u}) = 0$. Otherwise, the conclusion (b) contradicts (2.1). This completes our proof.

Remarks. 1. In case that $x \notin Fx$ implies $\inf_{u \in Fx} p(x, x-u) > 0$, then Theorem 3.2 generalizes Ding and Tan [DT1, Theorem 5] in several ways, which in turn generalizes Tan [T, Corollaries 4.5-4.8], Ko and Tan [KT, Theorem 3.3 and Corollary 3.5], Shih and Tan [ST, Theorem 10], Jiang [J, Theorem 3.3], and Browder [B2, Theroem 1; B1, Proposition 2].

2. As for Theorem 3.1, Theorem 3.2 also holds by replacing the inward sets by the corresponding outward sets. In this case, Theorem 3.2 generalizes Ding and Tan [DT1, Theorem 5′], Jiang [J, Corollary 3.4], and Browder [B2, Theorem 2].

3. For a normed vector space E with $p(x, y) = \|y\|$ for all $(x, y) \in X \times E$, Theorem 3.2 generalizes fixed point theorems due to Ding and Tan [DT1, Corollary 3], Shih and Tan [ST, Corollary 1], and Browder [B2, Corollaries 2 and 2′].

The following is a basis of best approximation or fixed point theorems for u.s.c. maps:

Theorem 3.3. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that $i_X : X \hookrightarrow E$ is continuous, $p : X \times E \to [0, \infty)$ a function such that, for each $x \in X$, $p(x, \cdot)$ is a continuous convex function on E. Suppose that $F : X \to k(E)$ satisfies the following:*

(3.1) *for each $y \in X$, $\{x \in X : \inf_{u \in Fx}[p(x, x-u) - p(x, y-u)] \leq 0\}$ is compactly closed in X;*

(3.2) *for each $x \in X \backslash K$ with $\inf_{u \in Fx} p(x, x-u) > 0$, there exists a $z \in \overline{I}_L(x)$ such that*

$$p(x, z-u) < p(x, x-u) \quad \text{for all} \quad u \in Fx.$$

Then either

(a) *there exist an $\hat{x} \in X \backslash K$ and a $\hat{u} \in F\hat{x}$ such that $p(\hat{x}, \hat{x} - \hat{u}) = 0$; or*

(b) *there exist an $\hat{x} \in K$ and a $\hat{u} \in F\hat{x}$ such that*

$$p(\hat{x}, \hat{x} - \hat{u}) \leq p(\hat{x}, y - \hat{u}) \quad \text{for all} \quad y \in \overline{I}_X(\hat{x}).$$

Proof. If $\inf_{u \in Fx} p(x, x-u) = 0$ for some $x \in X \backslash K$, then (a) holds. Suppose that $\inf_{u \in Fx} p(x, x-u) > 0$ for all $x \in X \backslash K$. Define a function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \inf_{u \in Fx}[p(x, x-u) - p(x, y-u)]$$

for $(x, y) \in X \times X$. We use Theorem 0 with $X = Y$, $T = 1_X$, $\gamma = 0$, and interchanging the roles of x and y.

(0.1) For ach $y \in X$, $\{x \in X : \phi(x, y) \leq 0\}$ is compactly closed by (3.1).

(0.2) For each $N \in \langle X \rangle$ and $x \in \operatorname{co} N$, $\min\{\phi(x, y) : y \in N\} \leq 0$.

Otherwise, there exist an $N = \{y_1, y_2, \ldots, y_n\} \in \langle X \rangle$ and an $\overline{x} = \sum_{i=1}^n r_i y_i \in \operatorname{co} N$ with $\sum_{i=1}^n r_i = 1$ and $r_i > 0$ for all i such that $\min\{\phi(\overline{x}, y) : y \in N\} > 0$; that is,

$$p(\overline{x}, \overline{x} - u) > p(\overline{x}, y - u)$$

for all $y \in N$ and $u \in F\overline{x}$. Since $p(\overline{x}, \cdot)$ is convex, for each $u \in F\overline{x}$, we have

$$\begin{aligned} p(\overline{x}, \overline{x} - u) &= p(\overline{x}, \sum_{i=1}^{n} r_i y_i - u) = p(\overline{x}, \sum_{i=1}^{n} r_i (y_i - u)) \\ &\leq \sum_{i=1}^{n} r_i p(\overline{x}, y_i - u) < p(\overline{x}, \overline{x} - u), \end{aligned}$$

which is a contradiction.

(0.3) Since L is c-compact, for each $N \in \langle X \rangle$, there exists an $L_N \in kc(X)$ containing N. Therefore, it suffices to show that, for each $x \in X \backslash K$, there exists a $y \in [x, L]$ satisfying $\phi(x, y) > 0$.

Suppose, on the contrary, that there is an $\overline{x} \in X \backslash K$ such that $\phi(\overline{x}, y) \leq 0$ or, equivalently

$$(*) \qquad p(\overline{x}, \overline{x} - u) \leq p(\overline{x}, y - u)$$

for all $y \in [\overline{x}, L]$ and all $u \in F\overline{x}$. However, since $\overline{x} \in X \backslash K$ and we have assumed $\inf_{u \in F\overline{x}} p(\overline{x}, \overline{x} - u) > 0$, by (3.2), there exists a $z_1 \in \overline{I}_L(\overline{x})$ such that

$$p(\overline{x}, z_1 - u) < p(\overline{x}, \overline{x} - u) \quad \text{for all} \quad u \in F\overline{x}.$$

Therefore, $z_1 \in \overline{I}_L(\overline{x}) \backslash [\overline{x}, L]$. Let us fix $u = \overline{u} \in F\overline{x}$. Since $p(\overline{x}, \cdot)$ is continuous, we may assume that $z_1 \in I_L(\overline{x}) \backslash [\overline{x}, L]$ and hence $z_1 = \overline{x} + \lambda(z - \overline{x})$ for some $z \in L$ and $\lambda > 1$ as $[\overline{x}, L]$ is convex. Note that

$$z = \frac{1}{\lambda} z_1 + (1 - \frac{1}{\lambda}) \overline{x} \in L \subset [\overline{x}, L]$$

and hence,

$$\begin{aligned} p(\overline{x}, z - \overline{u}) &= p(\overline{x}, \frac{1}{\lambda}(z_1 - \overline{u}) + (1 - \frac{1}{\lambda})(\overline{x} - \overline{u})) \\ &\leq \frac{1}{\lambda} p(\overline{x}, z_1 - \overline{u}) + (1 - \frac{1}{\lambda}) p(\overline{x}, \overline{x} - \overline{u}) \\ &< \frac{1}{\lambda} p(\overline{x}, \overline{x} - \overline{u}) + (1 - \frac{1}{\lambda}) p(\overline{x}, \overline{x} - \overline{u}) \\ &= p(\overline{x}, \overline{x} - \overline{u}), \end{aligned}$$

which contradicts $(*)$.

Now ϕ satisfies all of the requirements of Theorem 0. Therefore, there exists an $\hat{x} \in K$ such that $\phi(\hat{x}, y) \leq 0$; that is,

$$p(\hat{x}, \hat{x} - u) \leq p(\hat{x}, y - u)$$

for all $y \in X$ and all $u \in F\hat{x}$. Since $p(\hat{x}, \cdot)$ is continuous and $F\hat{x}$ is compact, there exists a $\hat{u} \in F\hat{x}$ such that $p(\hat{x}, \hat{x} - \hat{u}) = \inf_{u \in F\hat{x}} p(\hat{x}, \hat{x} - u)$, and hence

$$p(\hat{x}, \hat{x} - \hat{u}) \leq p(\hat{x}, y - \hat{u})$$

for all $y \in X$. Since $p(\hat{x}, \cdot)$ is convex and continuous, this inequality holds for all $y \in \overline{I}_X(\hat{x})$. This completes our proof.

Remarks. 1. As for Theorem 3.1, the inward sets in Theorem 3 can be replaced by the corresponding outward sets without affecting its conclusion.

2. Note that, in Theorem 3.3, the topology of the convex space X can be different from the relative topology with respect to E.

We also have the following:

Theorem 3.4. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that $i_X : X \hookrightarrow E$ is continuous, $p : X \times E \to [0, \infty)$, and $F : X \to k(E)$ such that*

(i) *p is continuous on $C \times D$ and F is u.s.c. on C for each $C \in k(X)$ and $D \in k(E)$; and*

(ii) *for each $x \in X$, $p(x, \cdot)$ is a convex function on E.*

Suppose that the following holds:

(4.1) *for each $x \in K$ with $\inf_{u \in Fx} p(x, x - u) > 0$, there exists a $z \in \overline{I}_X(x)$ such that $p(x, z - u) < p(x, x - u)$ for all $u \in Fx$; and*

(4.2) *for each $x \in X \backslash K$ with $\inf_{u \in Fx} p(x, x - u) > 0$, there exists a $z \in \overline{I}_L(x)$ such that $p(x, z - u) < p(x, x - u)$ for all $u \in Fx$.*

Then there exist an $\hat{x} \in X$ and a $\hat{u} \in F\hat{x}$ such that $p(\hat{x}, \hat{x} - \hat{u}) = 0$.

Proof. As in the proof of Theorem 3.3, consider the function $\phi : X \times X \to \overline{\mathbb{R}}$ given by

$$\phi(x, y) = \inf_{u \in Fx} [p(x, x - u) - p(x, y - u)]$$

for $(x, y) \in X \times X$. For each $C \in k(X)$, we have $F(C) \in k(E)$ since $F : X \to k(E)$ is u.s.c. on C. By the continuity of p, for each $y \in X$, the function $(x, u) \mapsto p(x, x - u) - p(x, y - u)$ is continuous on $C \times F(C)$. Therefore, by Lemma 1, for each $y \in X$, $x \mapsto \phi(x, y)$ is l.s.c. on each $C \in k(X)$. This imples (3.1). Moreover, (4.2) is the same as (3.2). Therefore, we can follow the proof of Theorem 3.3. Note that (4.1) assures that the $\hat{x} \in K$ in the conclusion (b) of Theorem 3.3 also satisfies the conclusion.

Remark. If $x \notin Fx$ implies $p(x, x - u) > 0$ for all $u \in Fx$, then Theorem 3.4 generalizes Ding and Tan [DT1, Theorem 6] in many ways, which in turn generalizes Shih and Tan [ST, Theorem 10] and Browder [B2, Theorem 1].

4. Best approximations and fixed points for continuous multifunctions

This section deals with very general best approximation or fixed point theorems for continuous multifunctions with compact convex values. Our main result follows from Theorem 3.1 and is applied to obtain fixed point theorems for a t.v.s. on which its topological dual E^* separates points.

The following is a general best approximation theorem for a continuous multifunction:

Theorem 4.1. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that i_X is continuous, $F : X \to kc(E)$, and $p : E \to [0, \infty)$ a continuous convex function. Suppose that*

(1.1) *for each $y \in X$, $\{x \in X : d_p(x, Fx) \le d_p(y, Fx)\}$ is compactly closed in X; and*

(1.2) *for each $x \in X \backslash K$ satisfying $d_p(x, Fx) > 0$, we have $d_p(\overline{I}_L(x), Fx) < d_p(x, Fx)$.*

Then either

(a) *there exists an $\overline{x} \in X \backslash K$ such that $d_p(\overline{x}, F\overline{x}) = 0$; or*

(b) *there exists an $\overline{x} \in K$ and a $\overline{u} \in F\overline{x}$ such that*

$$p(\overline{x} - \overline{u}) = d_p(\overline{I}_X(\overline{x}), F\overline{x}).$$

Note that, in Theorem 4.1, we use the notations

$$d_p(x, U) = \inf\{p(x - y) : y \in U\},$$
$$d_p(U, V) = \inf\{p(x - y) : x \in U,\ y \in V\}$$

for $x \in E$ and $U, V \subset E$.

Proof. Consider $p(x - y)$ instead of $p(x, x - y)$ in Theorem 3.1. Then all of the requirements of Theorem 3.1 are satisfied. Therefore, the conclusion follows.

Remarks. 1. As in Theorem 3.1, the inward sets in Theorem 4.1 can be replaced by the corresponding outward sets.

2. If we assume $d_p(x, Fx) > 0$ for each $x \in X \backslash K$ then case (a) in the conclusion can not occur.

3. If p in Theorem 4.1 is a continuous seminorm on E, then the $\overline{x} \in K$ in the conclusion (b) has the additional property:

$$\text{“}\overline{x} \in K \cap \operatorname{Bd} X \quad \text{whenever} \quad d_p(\overline{x}, F\overline{x}) > 0.\text{”}$$

In fact, if $\overline{x} \in \operatorname{Int} X$, then $\overline{x}$ is an internal point and hence $I_X(\overline{x}) = E$. So $d_p(\overline{x}, F\overline{x}) \le p(\overline{x} - \overline{u}) = d_p(\overline{I}_X(\overline{x}), F\overline{x}) = 0$ since $F\overline{x} \subset E$.

4. In (b), $\overline{u} \in F\overline{x}$ is so chosen that $p(\overline{x} - \overline{u}) = d_p(\overline{x}, F\overline{x})$. See the proof of Theorem 3.1.

Lemma 4. *Let X be a convex space, E a t.v.s. containing X as a subset, and $p : E \to [-\infty, \infty)$ a continuous function such that $p|_X : X \to [-\infty, \infty)$ is l.s.c. on each $C \in k(X)$.*

(i) *If $F : X \to 2^E$ is l.s.c. on C, then, for each $y \in X$, $x \mapsto d_p(y, Fx)$ is u.s.c. on C.*

(ii) *If $F : X \to k(E)$ is u.s.c. on C, then $x \mapsto d_p(x, Fx)$ is l.s.c. on C.*

(iii) *If $F : X \to k(E)$ is continuous on C, then, for each $y \in X$,*

$$\{x \in X : d_p(x, Fx) \leq d_p(y, Fx)\}$$

is compactly closed.

Proof. (i) For each $y \in X$, put $g(x, u) = p(y - u)$ for $(x, u) \in X \times E$. Then $g|_{C \times E}$ is u.s.c. and $F|_C$ is l.s.c. Now apply Lemma 2.

(ii) Let $g(x, u) = p(x - u)$ for $(x, u) \in X \times E$. Since $p|_X$ is l.s.c. on C and p is continuous on E, $g|_{C \times E}$ is l.s.c. Since $F|_C : C \to k(E)$ is u.s.c., we can apply Lemma 1.

(iii) For each $y \in X$, $x \mapsto d_p(x, Fx) - d_p(y, Fx)$ is l.s.c. on C.

From Theorem 4.1 and Lemma 4, we have the following generalized best approximation theorem:

Theorem 4.2. *Let X, K, L, E and F be the same as in Theorme 4.1, and $p : E \to [0, \infty)$ a continuous convex function. Suppose that*

(2.1) *$F : X \to kc(E)$ is continuous on each $C \in k(X)$; and*

(2.2) *for each $x \in X \backslash K$,*

$$d_p(\overline{I}_L(x), Fx) < d_p(x, Fx).$$

Then there exists a $u \in K$ such that

$$d_p(u, Fu) = d_p(\overline{I}_X(u), Fu).$$

Proof. We use Theorem 4.1. By Lemma 4, (2.1) implies (1.1). Note that (2.2) is a particular case of (1.2). Therefore, (a) or (b) of Theorem 4.1 holds. However, $d_p(x, Fx) > 0$ for all $x \in X \backslash K$ by (2.2) and hence (a) can not occur. Now (b) yields our conclusion.

Remarks. 1. The topologies of X and E are related by the requirement that i_X be continuous. If p is a continuous seminorm on E, then X may have a topology finer than the relative one w.r.t. E. If p is a weakly continuous seminorm on E, then X may have a topology finer than the relative one and E may have any topology finer than the weak one.

2. As in Theorem 4.1, if p is a continuous seminorm on E, then $u \in K \cap \operatorname{Bd} X$ whenever $d_p(u, Fu) > 0$.

3. Recently, Ding and Tan [DT2, Theorem 1] obtained a particular form of Theorem 4.2 for a t.v.s. E on which E^* separates points and a convex subset X of E having the relative weak topology. Actually they used the weak topology on E.

4. Moreover, Theorem 4.2 extends [DT3, Theorem 5] in many aspects. There, X has the relative topology of E, E^* separates points of E, L is a nonempty compact convex subset of X, p is a continuous seminorm on E, and F satisfies more stronger condition than (2.2). Similarly, for $X = L = K$, Theorem 4.2 extends [DT3, Theorem 7].

5. A number of authors obtained particular forms of Theorem 4.2 under the additional assumption that E is locally convex. See Fan [F2], Kapoor [K], Sehgal and Singh [SS, Theorem 1], Sehgal, Singh, and Smithson [SSS, Theorem], and Sehgal, Singh, and Whitfield [SSW, Theorem 4].

From Theorem 4.2 we obtain the following fixed point theorem for continuous multifunctions with compact convex values.

Theorem 4.3. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that i_X is continuous and E^* separates points of E, and $F : X \to kc(E)$ continuous on each $C \in k(X)$. Suppose that for each $p \in S(E, w)$ we have the following:*

(3.1) *for each $x \in X \backslash K$, we have*

$$d_p(\overline{I}_L(x), Fx) < d_p(x, Fx).$$

Then either

(a) *F has a fixed point in K; or*

(b) *there exist a $p \in S(E,w)$ and an $x \in K \cap \operatorname{Bd} X$ such that*

$$0 < d_p(x, Fx) = d_p(\overline{I}_X(x), Fx).$$

Proof. Suppose that F has no fixed point in K. Then for any $x \in K$, there exists an $h \in E^*$ separating x and $Fx \in kc(E)$; that is, $\operatorname{Re} h(x) > \sup \operatorname{Re} h(Fx)$. Hence, there is a $p_x \in S(E,w)$ with $p_x(y) = |\operatorname{Re} h(y)|$ for $y \in E$ such that $d_{p_x}(x, Fx) > 2\delta_x$ for some $\delta_x > 0$. Since F is u.s.c. and p_x is continuous on K, there exists an open neighborhood U_x of x in K such that $p_x(z - v) > \delta_x$ for all $z \in U_x$ and $v \in Fz$. Since $\{U_x\}_{x \in K}$ covers $K \in k(X)$, there exists a finite subset N of K such that $\{U_x\}_{x \in N}$ covers K. Let $p = \max\{p_x : x \in N\}$ and $\delta = \min\{\delta_x : x \in N\} > 0$. Then $p \in S(E,w)$ and $d_p(x, Fx) > \delta$ for all $x \in K$. For this $p \in S(E,w)$, there exists a $u \in K$ such that $d_p(u, Fu) = d_p(\overline{I}_X(u), Fu)$ by Theorem 4.2. Since $u \in K$, we have $d_p(u, Fu) > 0$ and hence $u \in K \cap \operatorname{Bd} X$. This completes our proof.

Remarks. 1. As before, the inward sets in Theorem 4.3 can be replaced by the corresponding outward sets, and X has any topology finer than the relative one and E has any topology finer than the weak one. This remark also works for Theorems 4.4-4.5.

2. Ding and Tan [DT2, Theorem 2 and Corollary 1] obtained particular forms of Theorem 4.3 for a convex subset X of E having the relative weak topology. Moreover, if X has the relative topology, p is a continuous seminorm, and F satisfies a stronger condition than (3.1), then Theorem 4.3 reduces to [DT3, Theorem 6].

3. Some other particular forms of Theorem 4.3 were due to Fan [F2], Reich [R1, Lemma 1.6], [R3, Theorem 1], Sehgal and Singh [SS, Corollary 2], and Park [P1, Theorem 3].

The following is a simple consequence of Theorem 4.3:

Theorem 4.4. *Let X, K, L, E, and F be the same as in Theorem 4.3. Suppose that for each $p \in S(E, w)$ we have the following:*

(4.1) *for each $x \in K \cap \operatorname{Bd} X$, we have $d_p(Fx, \overline{I}_X(x)) = 0$; and*

(4.2) *for each $x \in X \backslash K$, we have $d_p(Fx, \overline{I}_L(x)) < d_p(x, Fx)$.*

Then F has a fixed point in K.

Proof. Note that (4.1) implies the negation of (b) of Theorem 4.3, and (4.2) $\Longrightarrow$ (3.1). Therefore, the conclusion follows from Theorem 4.3.

Remark. Particular forms of Theorem 4.4 were due to Fan [F2, Corollary 6], Sehgal, Singh, and Whitfield [SSW, Corollary 6], Sehgal and Singh [SS, Corollary 2], Roux and Singh [RS, Theorems 5 and 6],and Ding and Tan [DT3, Theorem 8].

5. Best approximations and fixed points for u.s.c. multifunctions

In this section, we obtain general best approximation and fixed point theorems for u.s.c. maps with compact values. Our main result follows from Theorem 3.3 and is applied to obtain fixed point theorems.

The following is a basis of best approximation and fixed point theorems for compact-valued u.s.c. multifunction:

Theorem 5.1. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that i_X is continuous, $F : X \to k(E)$, $p : E \to [0, \infty)$ a continuous convex function. Suppose that*

(1.1) *for each $y \in X$, $\{x \in X : \inf_{u \in Fx}[p(x-u) - p(y-u)] \leq 0\}$ is compactly closed in X; and*

(1.2) *for each $x \in X \backslash K$ satisfying $d_p(x, Fx) > 0$, there exists a $z \in \overline{I}_L(x)$ such that*

$$p(z-u) < p(x-u) \quad \text{for all} \quad u \in Fx.$$

Then either

(a) *there exist an $\overline{x} \in X \backslash K$ and a $\overline{u} \in F\overline{x}$ such that $p(\overline{x} - \overline{u}) = 0$; or*

(b) *there exist an $\overline{x} \in K$ and a $\overline{u} \in F\overline{x}$ such that*

$$p(\overline{x} - \overline{u}) = d_p(\overline{I}_X(\overline{x}), \overline{u}).$$

Note that in Theorem 5.1, we use the notation

$$d_p(U, y) = \inf\{p(x - y) : x \in U\}$$

for $U \subset E$ and $y \in E$.

Proof. Consider $p(x - y)$ instead of $p(x, x - y)$ in Theorem 3.3. Then all of the requirements of Theorem 3.3 are satisfied. Therefore, the conclusion follows.

Remarks. 1. As usual, the inward sets in Theorem 5.1 can be replaced by the corresponding outward sets.

2. In (b), $\overline{u} \in F\overline{x}$ is so chosen that $p(\overline{x} - \overline{u}) = d_p(\overline{x}, F\overline{x})$. See the proof of Theorem 3.3.

3. If p in Theorem 5.1 is a continuous seminorm on E, then the $\overline{x} \in K$ in the conclusion (b) has the additional property:

$$\text{“}\overline{x} \in K \cap \operatorname{Bd} X \quad \text{whenever} \quad d_p(\overline{x}, F\overline{x}) > 0.\text{”}$$

From Theorem 5.1, we have the following generalized best approximation theorem:

Theorem 5.2. *Let X, K, L, E, and F be the same as in Theorem 5.1, and $p : E \to [0, \infty)$ a continuous convex function. Suppose that*

(2.1) *$F : X \to k(E)$ is u.s.c. on each $C \in k(X)$;*

(2.2) *for each $x \in X \backslash K$ with $d_p(x, Fx) > 0$, there exists a $z \in \overline{I}_L(x)$ such that*

$$p(z - u) < p(x - u) \quad \text{for all} \quad u \in Fx.$$

Then there exist an $\overline{x} \in X$ and a $\overline{u} \in F\overline{x}$ such that

$$p(\overline{x} - \overline{u}) = d_p(\overline{I}_X(\overline{x}), \overline{u}).$$

Proof. For each $y \in E$, define $g(x, u) = p(x - u) - p(y - u)$ for $(x, u) \in X \times E$. Since i_X is continuous on X and p is continuous on E, $g|_{C \times E}$ is continuous for each $C \in k(X)$. Since $F|_C : C \to k(E)$ is u.s.c., by Lemma 1,

$$x \mapsto \inf_{u \in Fx} [p(x - u) - p(y - u)]$$

is l.s.c. on each $C \in k(X)$. Therefore, (1.1) of Theorem 5.1 holds. Moreover, (2.2) is the same as (1.2). Hence, we have (a) or (b) of Theorem 5.1. Therefore, we have the conclusion.

Remarks. 1. The topologies of X and E are related as in Remark 1 of Theorem 4.2.

2. As in Theorem 5.1, if p is a continuous seminorm on E, then $\overline{x} \in K \cap \mathrm{Bd}\, X$ whenever $d_p(\overline{x}, F\overline{x}) > 0$.

3. Particular forms of Theorem 5.2 were due to Sehgal and Singh [SS, Theorem 1], Sehgal, Singh, and Smithson [SSS, Theorem], and Sehgal, Singh, and Whitfield [SSW, Theorem 4].

From Theorem 5.2 we obtain the following fixed point theorem for u.s.c. multifunction with compact convex values:

Theorem 5.3. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that E^* separates points of E such that i_X is continuous, and $F : X \to kc(E)$ u.s.c. on each $C \in k(X)$. Suppose that for each $p \in S(E, w)$ condition (2.2) holds. Then either*

(a) *F has a fixed point in K; or*

(b) *there exist a $p \in S(E, w)$, an $\overline{x} \in K \cap \mathrm{Bd}\, X$, and a $\overline{u} \in F\overline{x}$ such that*

$$0 < p(\overline{x} - \overline{u}) = d_p(\overline{I}_X(\overline{x}), \overline{u}).$$

Proof. Suppose that F has no fixed point in K. Then, as in the proof of Theorem 4.3, there exists a $p \in S(E, w)$ such that $d_p(x, Fx) > 0$ for all $x \in K$. For this $p \in S(E, w)$, there exists an $\overline{x} \in X$ and a $\overline{u} \in F\overline{x}$ such that $p(\overline{x} - \overline{u}) = d_p(\overline{I}_X(\overline{x}), \overline{u})$ by Theorem 5.2. By (2.2), we should have $\overline{x} \in K$ and hence $p(\overline{x} - \overline{u}) > 0$. Therefore, $\overline{x} \in K \cap \mathrm{Bd}\, X$. This completes our proof.

Remarks. 1. As usual, in Theorems 5.1-5.3, the inward sets can be replaced by the corresponding outward sets, and X may have a topology finer than the relative one and E may have a topology finer than the weak one.

2. Particular forms of Theorem 5.3 were due to Fan [F2, Theorme 1], Browder [B2, Corollaries 1 and 1′], Reich [R2, Theorem 3.1], Sehgal and Singh [SS, Corollary 1], Ha [H, Theorem 3], Park [P2, Theorem 3], and Ding and Tan [DT2, Theorem 4].

Considering the negation of (b), we obtain the following equivalent form of Theorem 5.3:

Theorem 5.4. *Under the hypothesis of Theorem 5.3, further assume that*

(4.1) *for each $p \in S(E, w)$, each $x \in K \cap \operatorname{Bd} X$ with $d_p(x, Fx) > 0$, and each $u \in Fx$, we have*

$$d_p(\overline{I}_X(x), u) < p(x - u).$$

Then F has a fixed point in K.

Remark. Particular forms of Theorem 5.4 for $X = K$ and a locally convex Hausdorff topological vector space E were due to Reich [R3, Theorem 2] and Ding and Tan [DT2, Theorem 5].

Theorem 5.5. *Under the hypothesis of Theorem 5.3, further assume that*

(5.1) *for each $p \in S(E, w)$, each $x \in \operatorname{Bd} X \backslash Fx$, and each $u \in Fx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that*

$$|\lambda| < 1 \quad \textit{and} \quad \lambda x + (1 - \lambda)u \in \overline{I}_X(x).$$

Then F has a fixed point.

Proof. We use Theorem 5.4. We show (5.1) $\Longrightarrow$ (4.1). For each $p \in S(E, w)$, each $x \in \mathrm{Bd}\, X \backslash Fx$ with $d_p(x, Fx) > 0$, and each $u \in Fx$, put $y = \lambda x + (1-\lambda)u \in \overline{I}_X(x)$. Then

$$d_p(\overline{I}_X(x), u) \leq p(y - u) = p(\lambda x - \lambda u) = |\lambda| p(x - u) < p(x - u)$$

since $|\lambda| < 1$ and $p(x - u) > 0$. Therefore, we have the conclusion.

Remark. For a locally convex Hausdorff topological spaces, particular forms of Theorem 5.5 were due to Fan [F2, Theorem 3], Reich [R2, Theorem 3.1], [R3, Theorem 2], Sehgal and Singh [SS, Corollary 2], Ha [H, Theorem 4], Park [P2, Theorem 4], and Ding and Tan [DT2, Theorem 6].

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