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FIXED POINTS OF APPROXIMABLE MAPS

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ABSTRACT. We present a simple proof of the Leray-Schauder type theorem for approximable multimaps given recently by Ben-El-Mechaiekh and Idzik. We apply this theorem to obtain a Schaefer type theorem, the Birkhoff-Kellogg type theorems, a Penot type theorem for non-self-maps, and quasi-variational inequalities, all related to compact closed approximable maps.

0. INTRODUCTION

Recently, Ben-El-Mechaiekh and Idzik [BI] derived a Leray-Schauder type theorem for approximable multimaps from a matching theorem of Ky Fan. We present, in this paper, a simple proof of their theorem using an earlier fixed point theorem due to Ben-El-Mechaiekh *et al.* Moreover, we apply their theorem to obtain a Schaefer type theorem, the Birkhoff-Kellogg type theorems, a Penot type theorem for non-self-multimaps, and quasi-variational inequalities with respect to compact closed approximable maps. Finally, we indicate that our method works also for other classes of maps including composites of acyclic maps.

A t.v.s. means a Hausdorff topological vector space. Int, Bd, $\overline{}$, and co denote the interior, boundary, closure, and convex hull, respectively.

For subsets X and Y of t.v.s. E and F, respectively, a *multimap* or *map* $\Phi : X \multimap Y$ is a function from X into the power set of Y with nonempty values. Φ is said to be *closed* if it has a closed graph $\mathrm{Gr}(\Phi) \subset X \times Y$, and *compact* if its range $\Phi(X)$ is contained in a compact subset of Y.

Given two open neighborhoods U and V of the origin 0 of E and F, respectively, a (U, V)-*approximative continuous selection* of Φ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (\Phi[(x + U) \cap X] + V) \cap Y \quad \text{for} \quad x \in X.$$

A map $\Phi : X \multimap Y$ is said to be *approximable* if its restriction $\Phi|_K$ to any compact subset K of X admits a (U, V)-approximative continuous selection for every U and V as above.

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For properties and examples of approximable maps, we refer to [BI] and references therein.

1. A Leray-Schauder type theorem

We begin with the following particular form of [P2, Theorem 3]:

Theorem 1. *Let X be a convex subset of a locally convex t.v.s. E and $\Phi : X \multimap X$ a compact closed approximable map. Then Φ has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in \Phi(\hat{x})$.*

A slightly particular version of Theorem 1 appeared in [Be2, Theorem 2.4], [BD2, Corollary 3.4], [BD3, Corollary 7.3]. In case X itself is compact, Theorem 1 is given in [BD2, Corollary 3.6], [BD3, Corollary 7.6].

From Theorem 1, we give a simple proof of the following theorem due to Ben-El-Mechaiekh and Idzik [BI]:

Theorem 2. *Let X be a closed subset of a locally convex t.v.s. E such that $0 \in \operatorname{Int} X$ and $\Phi : X \multimap E$ a compact closed approximable map. Then either*

(1) *Φ has a fixed point; or*
(2) *$\lambda x \in \Phi(x)$ for some $\lambda > 1$ and $x \in \operatorname{Bd} X$.*

Proof. Let $R \subset X$ be defined by

$$R = \{x \in X : x \in t\Phi(x) \quad \text{for some} \quad t \in [0,1]\},$$

which is nonempty since $0 \in R$. Moreover, it is closed since Φ is closed. Therefore, R is compact since Φ is compact.

Suppose that

(LS) $\Phi(y) \cap \{\lambda y : \lambda > 1\} = \emptyset \quad$ for all $\quad y \in \operatorname{Bd} X$.

Then $R \cap \operatorname{Bd} X = \emptyset$. Since X is completely regular, there exists a continuous function $r : X \to [0,1]$ such that $r(x) = 1$ for $x \in R$ and $r(x) = 0$ for $x \in \operatorname{Bd} X$.

Let $\Psi : E \multimap E$ be defined by

$$\Psi(x) = \begin{cases} r(x)\Phi(x) & \text{if} \quad x \in X, \\ \{0\} & \text{if} \quad x \notin X. \end{cases}$$

Since Φ is compact and closed, so is Ψ. Moreover, Ψ is approximable. In fact, for any compact subset K of X, if $s : K \to E$ is a (U,V)-approximative continuous selection of $\Phi|_K$, then $rs : K \to E$ is a (U,V)-approximative continuous selection of $\Psi|_K$. Therefore, Ψ has a fixed point $x \in E$ by Theorem 1. Since $0 \in \operatorname{Int} X \subset X$, if $x \notin X$ and $x \in \Psi(x) = \{0\}$, we have a contradiction. Hence, $x \in X$. Now $x \in \Psi(x)$ implies $x \in R$ and $r(x) = 1$. Therefore, $x \in \Phi(x)$. This completes our proof.

Remarks. 1. We followed the method of Schöneberg [Sö]. Theorem 2 includes [Be1, Theorem 5] and many others. See [BI], [P4].

2. In a sense, Theorem 2 is more general than Theorem 1. As Ben-El-Mechaiekh [Be1, p. 314] noted, Theorem 2 works for a self-map $\Phi : X \multimap X$ where X is star-shaped with the star-center $0 \in \operatorname{Int} X$. Moreover, Theorem 2 also works for a self-map $\Phi : \overline{X} \to \overline{X}$ where X is a shrinkable subset; that is, $[0,1)(\overline{X-p}) \subset \operatorname{Int}(X-p)$ for some $p \in X$. See Klee [K].

2. A Schaefer type theorem

From Theorem 2, we have the following :

Theorem 3. *Let E be a locally convex t.v.s. and $\Phi : E \multimap E$ a compact closed approximable map. Then either*

(1) *Φ has a fixed point; or*
(2) *the set $A = \{x \in E : x \in t\Phi(x)$ for some $t \in (0,1)\}$ is not bounded.*

Proof. Suppose that A is bounded. Let X be a bounded neighborhood of 0 such that $A \subset \operatorname{Int} X$. Then no $y \in \operatorname{Bd} X$ satisfies $\lambda y \in \Phi(y)$ for any $\lambda > 1$. Therefore, by Theorem 2, Φ has a fixed point in $\overline{X}$.

Remark. Theorem 3 was first obtained by Schaefer [Sc1, Sc2] for a completely continuous map $f : E \to E$ on a complete locally convex t.v.s. E.

3. The Birkhoff-Kellogg type theorems

As an application of Theorem 2, we have the following generalization of the Birkhoff-Kellogg theorem [BK].

Theorem 4. *Let X be a closed subset of a locally convex t.v.s. E such that $0 \in \operatorname{Int} X$, and $\Phi : X \multimap E$ a compact closed approximable map such that $\lambda\Phi(X) \cap X = \emptyset$ for some λ. Then $\Phi|_{\operatorname{Bd} X}$ has an eigenvalue; that is, $\mu x \in \Phi(x)$ for some $\mu \neq 0$ and $x \in \operatorname{Bd} X$.*

Proof. Note that $\lambda \neq 0$ and $\lambda\Phi : X \multimap E$ is a compact closed approximable map. Moreover, $\lambda\Phi$ has no fixed point. Therefore, by Theorem 2, there exist $x \in \operatorname{Bd} X$ and $\mu > 1$ such that $\mu x \in \lambda\Phi(x)$, whence we have $(\lambda^{-1}\mu)x \in \Phi(x)$, where $\lambda^{-1}\mu \neq 0$. This completes our proof.

Remark. If $\lambda > 0$ in Theorem 4, then $\Phi|_{\operatorname{Bd} X}$ has an invariant direction (a positive eigenvalue); that is, $\mu x \in \Phi(x)$ for some $\mu > 0$ and $x \in \operatorname{Bd} X$.

From Theorem 4, we obtain

Theorem 5. *Let S be the unit sphere of a normed vector space E of infinite dimension, and $\Phi : S \multimap E$ a compact closed approximable map such that $0 \notin \overline{\Phi(S)}$. Then Φ has an invariant direction.*

Proof. Since E is infinite dimensional, by the Dugundji extension theorem, there exists a retraction $r : E \to S$ such that $r(x) = x/\|x\|$ if $\|x\| \geq 1$ and $\|r(x)\| = 1$ if $\|x\| \leq 1$. Let $\Psi = \Phi r : E \multimap E$. Then Ψ is a compact closed approximable map. Let B be the closed unit ball. Then $\lambda\Psi(B) \cap B = \emptyset$ for some $\lambda > 0$ since $\Psi(B) \subset \overline{\Phi(S)}$ and $0 \notin \overline{\Phi(S)}$. Therefore, by Theorem 4 with $X = B$, $\Psi|_S$ has an eigenvalue. Since $\lambda > 0$, this eigenvalue is positive. This completes our proof.

Theorem 5 reduces immediately to the following fixed point theorem:

Theorem 6. *Let S be the unit sphere of a normed vector space E. Then E is of infinite dimension if and only if any compact closed approximable map $\Phi : S \multimap S$ has a fixed point.*

4. Fixed points of non-self-maps

Combining Theorems 1 and 2, we obtain the following fixed point theorem for approximable maps:

Theorem 7. *Let X be a closed convex subset of a locally convex t.v.s. E, and $\Phi : X \multimap E$ a compact closed approximable map. If $\Phi(\operatorname{Bd} X) \subset X$, then Φ has a fixed point.*

Proof. If $\operatorname{Int} X = \emptyset$, then $X = \operatorname{Bd} X$ and $\Phi : X \multimap X$ has a fixed point by Theorem 1. If $\operatorname{Int} X \neq \emptyset$, then we may assume $0 \in \operatorname{Int} X$. Now for each $x \in \operatorname{Bd} X$, $\Phi(x) \subset X$ implies $\Phi(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$ since X is shrinkable; that is, (LS) holds. Therefore, by Theorem 2, Φ has a fixed point.

Remark. For a compact closed map $\Phi : X \multimap E$ with convex values, Theorem 7 reduces to Penot [Pe, Proposition 1.4], which contains the particular case for a single-valued continuous map due to Brezis. See [Pe].

5. Quasi-variational or variational inequalities

From Theorem 7, we have the following quasi-variational inequality :

Theorem 8. *Let X be a closed convex subset of a locally convex t.v.s. E, Y a compact subset of E, and $f : X \times Y \to \mathbf{R}$ an u.s.c. function. Let $T : X \multimap Y$ be a closed map such that $T(\operatorname{Bd} X) \subset X \cap Y$. Suppose that*

(i) *the function M defined on X by*

$$M(x) = \sup_{y \in T(x)} f(x, y) \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(ii) *the map $\Phi : X \multimap Y$ defined on X by*

$$\Phi(x) = \{y \in T(x) : f(x, y) = M(x)\} \quad \textit{for} \quad x \in X$$

is approximable.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in T\hat{x} \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Proof. Note that the marginal function M in (i) is actually continuous since f is u.s.c. and T is a compact-valued u.s.c. map, by the well-known result of Berge [Br]. Now, each $\Phi(x)$ is nonempty. Moreover, Φ is a closed map. In fact, let $(x_\alpha, y_\alpha) \in \operatorname{Gr}(\Phi)$, the graph of Φ, and $(x_\alpha, y_\alpha) \to (x, y)$ in $X \times Y$. Then

$$\begin{aligned} f(x, y) &\geq \overline{\lim}_\alpha f(x_\alpha, y_\alpha) = \overline{\lim}_\alpha M(x_\alpha) \\ &\geq \underline{\lim}_\alpha M(x_\alpha) \geq M(x) \end{aligned}$$

and, since $\operatorname{Gr}(T)$ is closed in $X \times Y$, $y_\alpha \in T(x_\alpha)$ implies $y \in T(x)$. Hence $(x, y) \in \operatorname{Gr}(\Phi)$. Therefore, $\Phi : X \multimap E$ is a compact closed approximable map satisfying $\Phi(\operatorname{Bd} X) \subset T(\operatorname{Bd} X) \subset X$. Hence, by Theorem 7, Φ has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T\hat{x}$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. This completes our proof.

Remark. If $T : X \multimap Y \subset X$, then we can obtain Theorem 8 from Theorem 1 without assuming the closedness of X. In this case, Theorem 8 is actually equivalent to Theorem 1. Moreover, in this case, Theorem 8 extends Takahashi [T, Theorem 4], which was applied to prove Fan's generalizations of fixed point theorems of Schauder and Tychonoff.

The following is an immediate consequence of Theorem 8.

Theorem 9. *Let X be a compact convex subset of a locally convex t.v.s., $f : X \times X \to \mathbf{R}$ a continuous function such that all of the sets*

$$\{y \in X : f(x,y) = \inf_{y\in X} f(x,y)\}$$

for $x \in X$ are (1) *convex,* (2) *contractible,* (3) *decomposable, or* (4) *∞-proximally connected. Then there exists an $\hat{x} \in X$ such that*

$$f(\hat{x},\hat{x}) \leq f(\hat{x},y) \quad \textit{for all} \quad y \in X.$$

Proof. In any case (1)-(4), the map $\Phi : X \multimap X$ defined by

$$\Phi(x) = \{y \in X : f(x,y) = \inf_{y\in X} f(x,y)\} \quad \text{for} \quad x \in X$$

is a compact closed approximable map. See [BI]. Now, the conclusion follows from Theorem 8 by putting $X = Y$ and $T(x) = X$ for all $x \in X$.

Remark. As in [PC], Theorem 9 can be used to obtain variational or variational-like inequalities due to Hartman-Stampacchia, Browder, Lions-Stampacchia, Mosco, Juberg-Karamardian, Park, Karamardian, Parida-Sahoo-Kumar, Behera-Panda, and Siddiqi-Khaliq-Ansari. For the literature, see [PC].

Finally the approximable map in Theorem 1 can be replaced by an acyclic map [P1, Theorem 7(iii)], a composite of acyclic maps [PSW, Theorem 2(iii)], or a composite of admissible maps [P2, Theorem 3(iii)]. Therefore, following our method, Theorems 2-9 also hold for these classes of maps. In fact, we already have Theorem 8 for acyclic maps [P3, Theorem 2] and for admissible maps [PC, Theorem 2]. Hence, as was noted in [BI], the set in Theorem 9 can be acyclic.

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