THE IDZIK TYPE QUASI-VARIATIONAL INEQUALITIES AND NONCOMPACT OPTIMIZATION PROBLEMS

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Abstract. A quasi-variational inequality theorem equivalent to the Idzik fixed point theorem is obtained. Partial generalizations to condensing maps are also obtained. Those are applied to give simple and unified proofs of a number of well-known variational inequalities of the Hartman-Stampacchia-Browder type, and to solve a noncompact infinite optimization problem, which leads us to a generalized Nash equilibrium theorem.

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0. Introduction

Around 1930, Schauder conjectured that every compact convex subset of a topological vector space would have the fixed point property. During the last three decades, this old conjecture was intensively examined by many mathematicians. However, until now the conjecture is not resolved yet.

In an attempt to resolve the conjecture, Idzik [I3], in 1988, obtained a very remarkable fixed point theorem for not-necessarily locally convex topological vector spaces. His theorem is one of the most general results and extends a large number of known theorems.

In the present paper, we first obtain a quasi-variational inequality equivalent to the Idzik theorem, and then a partial generalization to condensing maps in the forms of a fixed point theorem and a quasi-variational inequality. Our new results are applied to give simple and unified proofs of the known variational inequalities of the Hartman-Stampacchia-Browder type. Finally, as an application of the Idzik theorem, we obtain a solution of a noncompact infinite optimization theorem, which leads us to a generalization of the Nash equilibrium theorem.

1. Compact maps

Let E be a real Hausdorff topological vector space (in short, a *t.v.s.*). A set $B \subset E$ is said to be *convexly totally bounded* (c.t.b.) whenever for every neighborhood V of $0 \in E$, there exist a finite subset $\{x_i : i \in I\} \subset E$ and a finite family of convex sets $\{C_i : i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i : i \in I\}$. See Idzik [I3] and Weber [W].

Let δ be the fundamental system of neighborhoods of the origin 0 in E. We recall that a set $K \subset E$ is *locally convex* if for every $x \in K$ and every $V \in \delta$ there exists $U \in \delta$ such that $\mathrm{co}((x + U) \cap K) \subset x + V$. We say that $K \subset E$ is *of Z type* if for every $V \in \delta$ there exists $U \in \delta$ such that $\mathrm{co}(U \cap (K - K)) \subset V$. See [H].

The following are known [I3]:

(1) Every compact subset of a locally convex t.v.s. is c.t.b.

(2) If E is locally convex, then every subset $K \subset E$ is of Z type and is a locally convex set.

(3) If $K \subset E$ is a compact subset which is either locally onvex or of Z type, then it is c.t.b.

(4) If the topological dual E^* of E separates points, then every compact convex subset of E is c.t.b. [W].

In this paper, a *multifunction* or *map* $T : X \multimap Y$ is nonempty valued.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) iff for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : Tx \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) iff for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* iff it is u.s.c. and l.s.c.

We begin with the following particular form of Idzik's theorem [I3, Theorem 4.3]:

Theorem 0. *Let X be a nonempty convex subset of a t.v.s. E and $T : X \multimap X$ a closed map with convex values. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Recall that T is said to be *closed* if its graph $\mathrm{Gr}(T)$ is closed in $X \times X$ and *compact* if $\overline{T(X)}$ is a compact subset of X. Note that every u.s.c. map T is closed.

Theorem 0 generalizes earlier results due to Zima, Rzepecki, Himmelberg, and Hadžić. For references, see [I3] or [H].

Recall that a real-valued function $g : X \to \mathbf{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] iff $\{x \in X : g(x) > r\}$ [resp. $\{x \in X : g(x) < r\}$ is open for each $r \in \mathbf{R}$. If X is a convex set in a vector space, then $g : X \to \mathbf{R}$ is *quasiconcave* [resp. *quasiconvex*] iff $\{x \in X : g(x) > r\}$ [resp. $\{x \in X : g(x) < r\}$] is convex for each $r \in \mathbf{R}$.

The following form of quasi-variational inequality is equivalent to Theorem 0:

Theorem 1. *Let X be a nonempty convex subset of a t.v.s. E, $f : X \times X \to \mathbf{R}$ an u.s.c. function, and $S : X \multimap X$ a closed compact map such that $\overline{S(X)}$ is c.t.b. Suppose that*

(1) *the function M defined on X by*

$$M(x) = \sup_{y \in S(x)} f(x, y) \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$\{y \in S(x) : f(x, y) = M(x)\}$$

is convex.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in S(\hat{x}) \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Proof. Note that the marginal function M in (1) is actually continuous by the well-known theorem of Berge [B, Theorem 2, Section 3, Chapter VI]. Define a map $T : X \multimap X$ by

$$T(x) = \{y \in S(x) : f(x, y) = M(x)\}$$

for $x \in X$. Note that each $T(x)$ is nonempty and convex by (2). We show that the graph $\mathrm{Gr}(T)$ is closed in $X \times X$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f(x, y) &\geq \overline{\lim_{\alpha}} f(x_\alpha, y_\alpha) = \overline{\lim_{\alpha}} M(x_\alpha) \\ &\geq \underline{\lim_{\alpha}} M(x_\alpha) \geq M(x) \end{aligned}$$

and, since $\mathrm{Gr}(S)$ is closed in $X \times X$, $y_\alpha \in S(x_\alpha)$ implies $y \in S(x)$. Hence $(x, y) \in \mathrm{Gr}(T)$. Since $\overline{T(X)} \subset \overline{S(X)}$ and S is compact, $\overline{T(X)}$ is a compact c.t.b. subset of X. Therefore, by Theorem 0, T has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in S(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. This completes our proof.

Remarks. 1. If $f(x,y) \equiv 0$ for all $x, y \in X$, then Theorem 1 reduces to Theorem 0. If f and S are continuous, then condition (1) holds by the theorem of Berge [B].

2. For a locally convex t.v.s. E, particular forms of Theorem 1 were obtained by Takahashi [T, Theorem 4] and Im and Kim [IK, Theorem 1]. Those authors applied their results to best approximation problems and optimization problems, respectively. See also Park [P2] and Park and Chen [PC].

2. Φ-condensing maps

In this section, we show that Theorem 1 also holds for condensing maps, whenever the domain X is closed, instead of compact maps.

Let E be a t.v.s. and C a lattice with a least element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ iff $\overline{X}$ is compact;

(2) $\Phi(\overline{\mathrm{co}}\, X) = \Phi(X)$, where $\overline{\mathrm{co}}$ denotes the convex closure of X; and

(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

It follows that $X \subset Y$ implies $\Phi(X) \leq \Phi(Y)$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined either in terms of a family of seminorms or a norm. For details, see [PF].

If $T : X \multimap E$, $X \subset E$, then T is called Φ-*condensing* provided that if $D \subset X$ and $\Phi(D) \leq \Phi(T(D))$, then $\overline{D}$ is compact; that is, $\Phi(D) = 0$.

Every map defined on a compact set is Φ-condensing. Note also that every compact map is Φ-condensing. See [MTY].

The following is recently due to Mehta, Tan, and Yuan [MTY, Lemma 1] for a locally convex t.v.s., but the proof works also for any t.v.s.

Lemma. *Let X be a nonempty closed convex subset of a t.v.s. E and Φ a measure of noncompactness on E. If $T : X \multimap X$ is Φ-condensing, then there exists a nonempty compact convex subset K of X such that $T(K) \subset K$.*

From Theorem 0 and Lemma, we obtain the following fixed point theorem for Φ-condensing maps:

Theorem 2. *Let X be a nonempty closed convex subset of a t.v.s. E and Φ a measure of noncompactness on E. If $T : X \multimap X$ is a closed Φ-condensing map with convex values such that $\overline{T(X)}$ is a c.t.b. subset of X, then T has a fixed point.*

Proof. By Lemma, there exists a nonempty compact convex subset K of X such that $T(K) \subset K$. Then $T|_K$ is a closed map with convex values such that $\overline{T(K)} \subset \overline{T(X)}$. Since $\overline{T(K)}$ is a compact c.t.b. subset of K, $T|_K$ has a fixed point.

Theorem 2 has the following equivalent formulation of a quasi-variational inequality:

Theorem 3. *Let X be a nonempty closed convex subset of a t.v.s. E, Φ a measure of noncompactness on E, $f : X \times X \to \mathbf{R}$ an u.s.c. function, and $S : X \multimap X$ a closed Φ-condensing map such that $\overline{S(X)}$ is a c.t.b. subset of X. Suppose that conditions (1) and (2) of Theorem 1 hold.*

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in S(\hat{x}) \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Proof. Define a map $T : X \multimap X$ as in the proof of Theorem 1. Then T is a closed map with convex values such that $\overline{T(X)}$ is a c.t.b. subset of X. We show that T is also a Φ-condensing. In fact, suppose that $D \subset X$ and $\Phi(D) \leq \Phi(T(D))$. Then $\Phi(D) \leq \Phi(T(D)) \leq \Phi(S(D))$. Since S is Φ-condensing, we have $\Phi(D) = 0$ and hence T is Φ-condensing. Therefore, by Theorem 2, T has a fixed point. This completes our proof.

Remark. If $f(x, y) \equiv 0$ for all $x, y \in X$, then Theorem 3 reduces to Theorem 2.

3. Applications to variational inequalities

In this section, we apply Theorems 1 and 3 to give simple proofs of the variational inequalities of the Hartman-Stampacchia-Browder type.

(i) Hartman and Stampacchia [HS, Lemma 3.1]: *Let K be a compact convex set in $\mathbf{R}^n$ and $B : K \to \mathbf{R}^n$ a continuous map. Then there exists $u_0 \in K$ such that*

$$\langle B(u_0), v - u_0 \rangle \geq 0 \quad \textit{for all} \quad v \in K.$$

Put $X = K$, $f(x, y) = \langle B(x), -y \rangle$, $S(x) = K$ for $x, y \in K$, and apply Theorem 1 or 3.

(ii) Browder [B1, Theorem 3; B2, Theorem 2]: *Let E be a t.v.s. on which its topological dual E^* is equipped with a topology such that the pairing $\langle\ ,\ \rangle : E^* \times E \to \mathbf{R}$ is continuous. Let K be a compact convex c.t.b. subset of E, and $T : K \to E^*$ continuous. Then there exists a $u_0 \in K$ such that*

$$\langle T(u_0), v - u_0 \rangle \geq 0 \quad \textit{for all} \quad v \in K.$$

Apply Theorem 1 as in (i).

(iii) Lions and Stampacchia [LS], Stampacchia [S], and Mosco [M, p.94]: *Let V be an inner product space, X a compact convex subset of V, and $a : V \times V \to \mathbf{R}$ a continuous bilinear form on V. Then for every $v' \in V^*$, there exists a (unique) vector $u \in X$ such that*

$$a(u, u - w) \leq \langle v', u - w \rangle \quad \textit{for all} \quad w \in X.$$

Put $X = K$, $V = E$, $S(x) = K$ for $x \in X$,

$$f(u, w) = a(u, -w) - \langle v', -w \rangle \quad \text{for} \quad u, w \in X,$$

and apply Theorem 1.

(iv) Karamardian [K, Lemma 3.2]: *Let X be a compact convex c.t.b. subset of a t.v.s. E, F a topological space, $g : X \to F$ a function, and $\psi : X \times F \to \mathbf{R}$ a function. If for each $y \in F$, $\psi(\cdot, y)$ is quasiconvex on X and the function $(u, v) \mapsto \psi(u, g(v))$ is continuous on $X \times X$, then there exists an $\overline{x} \in X$ such that*

$$\psi(\overline{x}, g(\overline{v})) \leq \psi(x, g(\overline{x})) \quad \textit{for all} \quad x \in X.$$

Put $S(x) = X$, $f(x, y) = -\psi(y, g(x))$ for $x, y \in D$, and apply Theorem 1.

Note that Karamardian [K] applied (iv) to obtain a variational inequality (v) in below, Fan's best approximation theorem, and a solution of the generalized complementarity theorem [K, Theorem 3.1].

(v) Karamardian [K, Corollary 3.1], Juberg and Karamardian [JK, Lemma], Park [P1, Corollary 1.3]: *Let X be a compact convex c.t.b. subset of a t.v.s. E, F a topological space, and $\langle\ ,\ \rangle : F \times E \to R$ a function which is linear in the second variable. Suppose that $g : X \to F$ is a function such that $(x, y) \mapsto \langle g(x), y \rangle$ is continuous on $X \times E$. Then there exists an $\overline{x} \in X$ such that*

$$\langle g(\overline{x}), y - \overline{x} \rangle \geq 0 \quad \textit{for all} \quad y \in X.$$

Put $S(x) = X$, $f(x, y) = \langle g(x), -y \rangle$ for $x, y \in X$, and apply Theorem 1.

(vi) Parida, Sahoo, and Kumar [PSK, Theorem 3.1], Behera and Panda [BP, Theorem 2.2], Siddiqi, Khaliq, and Ansari [SKA]: *Let X be a compact convex c.t.b. subset of a t.v.s. E on which E^* is equipped with a topology such that the pairing $\langle\ ,\ \rangle : E^* \times E \to \mathbf{R}$ is continuous, $T : X \to E^*$ and $\theta : K \times K \to E$ continuous maps such that*

(1) $\langle T(y), \theta(y, y) \rangle \geq 0$ *for all $y \in X$; and*

(2) *for each $y \in X$, the function $\langle Ty, \theta(\cdot, y) \rangle : X \to \mathbf{R}$ is quasiconvex.*

Then there exists an $x_0 \in X$ such that

$$\langle T(x_0), \theta(y, x_0) \rangle \geq 0 \quad \textit{for all} \quad y \in X.$$

Put $S(x) = X$, $f(x, y) = -\langle T(x), \theta(y, x) \rangle$ for $x, y \in X$, and apply Theorem 1.

Remarks. 1. Note that (ii) and (iv)-(vi) are stated in more general forms than the originally given ones.

2. In the frame of the KKM theory, some of (i)-(vi) can be obtained without assuming the property of c.t.b. However, in this section, we wanted to show the applicability of the Idzik theorem.

4. A noncompact infinite optimization problem

As an another application of the Idzik theorem, in this section, we consider a noncompact infinite optimization problem for a non-locally convex t.v.s.

Let I be any index set and, for each $i \in I$, E_i be a t.v.s. For subsets $X_i \subset E_i$, we use the notation

$$X = \prod_{i\in I} X_i \quad \text{and} \quad X^i = \prod_{j\in I, j\neq i} X_j.$$

For each $x \in X$, $x_i \in X_i$ denotes its ith coordinate and $x^i \in X^i$ the projection of x in X^i. Let $x = (x^i, x_i)$.

From Theorem 0, we deduce the following:

Theorem 4. *Let I be an index set, and for each $i \in I$, X_i be a convex subset of a t.v.s. E_i, D_i be a nonempty compact subsets of X_i such that $D = \prod_{i\in I} D_i$ is a c.t.v. subset of $E = \prod_{i\in I} E_i$. For each $i \in I$, let $f_i : X = \prod_{i\in I} X_i \to \mathbf{R}$ be an u.s.c. function, and $S_i : X^i \multimap D_i$ a closed map such that*

(1) *the function M_i defined on X^i by*

$$M_i(x^i) = \sup_{y\in S_i(x^i)} f_i(x^i, y) \quad \textit{for} \quad x^i \in X^i$$

is l.s.c.; and

(2) *for each $x^i \in X^i$, the set*

$$T_i(x^i) = \{y \in S_i(x^i) : f_i(x^i, y) = M_i(x^i)\}$$

is convex.

Then there exists an $\overline{x} \in D$ *such that for each* $i \in I$*,*

$$\overline{x}_i \in S_i(\overline{x}^i) \quad \textit{and} \quad f_i(\overline{x}^i, \overline{x}_i) = M_i(\overline{x}^i).$$

Proof. As in the proof of Theorem 1, the map $T_i : X^i \multimap D_i$ is a closed compact map. Define $T : X \multimap D$ by

$$T(x) = \prod_{i \in I} T_i(x^i) \quad \text{for} \quad x \in X.$$

Then T is also a closed compact map with convex values by [F, Lemma 3] and the assumption (2). Since $\overline{T(X)} \subset D$ is c.t.v., by Theorem 0, T has a fixed point $\overline{x} \in D$; that is, $\overline{x}_i \in T_i(\overline{x}^i) \subset S_i(\overline{x}^i)$ and $f_i(\overline{x}^i, \overline{x}_i) = M_i(\overline{x}^i)$ for all $i \in I$. This completes our proof.

Remarks. 1. If each E_i is locally convex and each f_i and S_i are continuous, then Theorem 4 reduces to Idzik [I1, Theorem 7], which includes later works of Im and Kim [IK, Theorem 2] and Kaczynski and Zeidan [KZ]. In [I2, Theorem 7], a related result has been proved for a general t.v.s.

2. Instead of the compactness of S_i, as in Theorem 3, we may obtain a result for Φ-condensing maps S_i.

From Theorem 4, we obtain the following infinite version of the Nash equilibrium theorem:

Theorem 5. *Let* I *be an index set, and for each* $i \in I$*,* X_i *be a nonempty compact convex subset of a t.v.s.* E_i *such that* $X = \prod_{i \in I} X_i$ *is a c.t.b. subset of* $E = \prod_{i \in I} E_i$*. For each* $i \in I$*, let* $f_i : X \to \mathbf{R}$ *be a continuous function such that for each given point* $x^i \in X^i$*,* $x_i \mapsto f(x^i, x_i)$ *is a quasiconcave function on* X_i*. Then there exists an* $\overline{x} \in X$ *such that*

$$f_i(\overline{x}) = f_i(\overline{x}^i, \overline{x}_i) = \max_{y_i \in X_i} f_i(\overline{x}^i, y_i) \quad \textit{for each} \quad i \in I.$$

Proof. Let $D_i = X_i$ and $S_i(x^i) = X_i$ for each $x^i \in X^i$ and each $i \in I$. Then S_i is a continuous map. Since each f_i and S_i are continuous with compact values, condition (1) of Theorem 4 is satisfied by the theorem of Berge [B]. Note that condition (2) holds by the quasiconcavity of f_i. Therefore, the conclusion follows from Theorem 4 immediately.

Remarks. 1. Note that Ma [M, Theorem 4] already established Theorem 5 without assuming that X is c.t.b. A generalization of Ma's theorem was given in [I2, Theorem 7].

2. Nash's original theorem is the case E_i are Euclidean spaces and I is finite. See [N].

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