Acta Math. Hungar.
70 (3) (1996), 193–197.

SET-VALUED NONEXPANSIVE MAPS SATISFYING THE LERAY–SCHAUDER CONDITION

S. PARK (Seoul)*

In [5], Gulevich obtained the following:

THEOREM. *Let H be a Hilbert space, D a subset of H, $f : D \to H$ a nonexpansive map, and A a nonempty bounded closed subset of H such that $\overline{\mathrm{co}}\, A \subset D$. If $f(\mathrm{Bd}\, A) \subset A$, then f has a fixed point in $\overline{\mathrm{co}}\, A$.*

This is the basic theorem of Gulevich [5, Theorem 1], from which he deduced many results. At the end of his paper he raised some problems. The first one was whether his theorem holds for uniformly convex Banach spaces. We showed in our previous work [12] that if A is star-shaped then it is affirmative and we can adopt more general boundary conditions.

The third problem of Gulevich was whether his theorem holds for set-valued maps. In the present paper, we show that if A is star-shaped, then we have set-valued versions of results of [12], from which we can deduce a Gulevich type theorem for uniformly convex Banach spaces. Moreover, we adopt more general boundary conditions such as the weakly inwardness or the Leray–Schauder condition than the Rothe condition in Gulevich's theorem.

Let T be a map of a complete metric space (M, d) which take values in the family $bc(M)$ of nonempty bounded closed subsets of M where this family is given the Hausdorff metric H. T is called a *contraction* if there exists an $\alpha \in (0, 1)$ such that

$$H(Tx, Ty) \leqq \alpha d(x, y) \quad \text{for all} \quad x, y \in M,$$

and a *nonexpansive map* if

$$H(Tx, Ty) \leqq d(x, y) \quad \text{for all} \quad x, y \in M.$$

In a Banach space X, let $k(X)$ denote the family of all nonempty compact subsets of X, $kc(X)$ all nonempty compact convex subsets, and $ka(X)$ all compact acyclic subsets. Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

Let Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, respectively.

* Supported in part by KOSEF-951-0102-006-2.

For a subset K of a vector space X and a point $x \in X$, the *inward set* of K at x is defined by

$$I_K(x) = \{x + c(y - x) : y \in K, \quad c \geqq 1\}.$$

THEOREM 0. *Let X be a Banach space, K a closed subset of X, and $T : K \to bc(X)$ a contraction satisfying one of the following conditions:*
(I) $T(\operatorname{Bd} K) \subset K$.
(II) $T : K \to k(X)$ *and*

$$Tx \subset \bar{I}_K(x) \quad \text{for all } x \in \operatorname{Bd} K.$$

(III) K *bounded convex,* $T : K \to kc(X)$, *and*

$$Tx \cap \bar{I}_K(x) \neq \emptyset \quad \text{for all } x \in \operatorname{Bd} K.$$

(IV) $T : K \to ka(X)$, $0 \in \operatorname{Int} K$, $T(K)$ *bounded, and*

$$Tx \cap \{\lambda x : \lambda > 1\} = \emptyset \quad \text{for all } x \in \operatorname{Bd} K.$$

Then T has a fixed point.

Note that Theorem 0(I), (II), (III), and (IV) are particular forms of Assad and Kirk [1, Theorem 1], Zhang [14, Theorem 3.3], Deimling [4, Theorem 1], and Park [11, Theorem 7], respectively.

For an $x \in K$, define the geometric estimator

$$G(x, Tx) = \{z \in K : \|z - x\| \geqq d(z, Tx)\},$$

where $d(z, Tx) = \inf_{y \in Tx} \|z - y\|$. We say that K is (KR)-*bounded* if, for some bounded set $A \subset K$, the set

$$G(A) = \bigcap_{u \in A} G(u, Tu)$$

is either empty or bounded. For single-valued case, this concept is due to Kirk and Ray [7].

From Theorem 0, we have the following:

THEOREM 1. *Let X be a Banach space, K a closed subset of X, and $T : K \to k(X)$ a nonexpansive map satisfying one of conditions* (I)–(IV) *of Theorem* 0. *Suppose that K is star-shaped for cases* (I) *and* (II), *and that K is (KR)-bounded for cases* (I), (II), *and* (IV). *Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in K such that $y_n \in Tx_n$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

PROOF. For cases (I)–(III) we may without loss of generality assume that 0 is the star-shape center. For $\alpha \in (0,1)$, define $T_\alpha : K \to k(X)$ by $T_\alpha x$

$= \alpha Tx$ for $x \in K$. Then clearly T_α is a contraction. We show that T_α satisfies one of (I)–(IV) in Theorem 0 as follows:

(I) For each $x \in \operatorname{Bd} K$, $T_\alpha x = \alpha Tx \subset \alpha K \subset K$ since K is star-shaped with center 0.

(II) For each $x \in \operatorname{Bd} K$, we have $Tx \subset \overline{I}_K(x)$. Therefore, $T_\alpha x = \alpha Tx \subset \alpha \overline{I}_K(x) \subset \overline{I}_K(x)$ since $I_K(x)$ is a star-shaped set with center 0. See Zhang [14, Theorem 1.2].

(III) Since $Tx \cap \overline{I}_K(x) \neq \emptyset$ for $x \in \operatorname{Bd} K$, we have $T_\alpha x \cap \overline{I}_K(x) \supset T_\alpha x \cap \alpha \overline{I}_K(x) = \alpha Tx \cap \alpha \overline{I}_K(x) \supset \alpha\big(Tx \cap \overline{I}_K(x)\big) \neq \emptyset$.

(IV) Suppose that $\lambda x \in T_\alpha x$ for some $x \in \operatorname{Bd} X$ and $\lambda \nabla 1$. Then $\lambda x \in T_\alpha x = \alpha Tx$ and hence $(\alpha^{-1}\lambda)x \in Tx$ and $\alpha^{-1}\lambda \nabla 1$. This violates our assumption.

Therefore, for each case, the requirements of Theorem 0 are all satisfied for T_α. Hence T_α has a fixed point $x_\alpha \in K$.

Suppose that the set $\{x_\alpha : \alpha \in (0,1)\}$ is not bounded. Then it is possible to choose $\alpha \in (0,1)$ so that

$$\sup_{z \in T(A)} \|z\| \leqq \inf_{u \in A} \|x_\alpha - u\|$$

for some bounded set $A \subset K$ such that $G(A)$ is empty or bounded, and in addition, if $G(A) \neq \emptyset$, then α may also be chosen so that

$$\|x_\alpha\| > \sup_{x \in G(A)} \|x\|.$$

Moreover, for each $u \in A$, there exists a $z_u \in Tu$ such that $d(x_\alpha, \alpha Tu) = \|x_\alpha - \alpha z_u\|$ since Tu is compact. Then

$$\begin{aligned} d(x_\alpha, Tu) &\leqq \|x_\alpha - z_u\| \leqq \|x_\alpha - \alpha z_u\| + (1-\alpha)\|z_u\| \\ &\leqq d(x_\alpha, \alpha Tu) + (1-\alpha) \inf_{u \in A} \|x_\alpha - u\| \\ &\leqq \alpha H(Tx_\alpha, Tu) + (1-\alpha)\|x_\alpha - u\| \\ &\leqq \alpha\|x_\alpha - u\| + (1-\alpha)\|x_\alpha - u\| = \|x_\alpha - u\|. \end{aligned}$$

This implies $x_\alpha \in G(A)$, which leads to a contradiction. Thus $M = \sup\{\|x_\alpha\| : \alpha \in (0,1)\} < \infty$. Let $y_\alpha = \alpha^{-1}x_\alpha \in \alpha^{-1}T_\alpha x_\alpha = Tx_\alpha$. Then we have

$$\|x_\alpha - y_\alpha\| = (\alpha^{-1} - 1)\|x_\alpha\| \leqq (\alpha^{-1} - 1)M,$$

yielding $\|x_\alpha - y_\alpha\| \to 0$ as $\alpha \to 1$. This completes our proof.

Note that for a single-valued map $T : K \to X$, Theorem 1 reduces to Park [12, Theorem 1], which extends Kirk and Ray [7, Theorem 2.3]. Note also that Theorem 1(II) generalizes Canetti et al. [3, Theorem 2], where K

is assumed to be convex. Therefore, [3, Corollaries 4-8] are all consequences of Theorem 1(II).

A map $T : K \to 2^X$ is said to be *demi-closed* [2, 13] if, whenever $x_n \to x$ weakly while $y_n \to y$ strongly and $y_n \in Tx_n$, then $y \in Tx$.

A Banach space X is said to satisfy *Opial's condition* [10] if, whenever a sequence $\{x_n\}$ converges weakly to $x_0 \in X$, then

$$\liminf_{n\to\infty} \|x_n - x\| > \liminf_{n\to\infty} \|x_n - x_0\|$$

for all $x \in X$, $x \neq x_0$. Lami Dozo [8] showed that if C is a weakly compact subset of a Banach space X satisfying this condition and $F : C \to 2^X$ is nonexpansive, then $I - T$ is demi-closed. The convexity of C is not necessary. See Zhang [14].

From Theorem 1, we have the following:

THEOREM 2. *Let X be a Banach space, K a weakly compact subset of X, and $T : K \to k(X)$ a nonexpansive map satisfying one of the conditions* (I)–(IV) *of Theorem* 0. *Suppose that K is star-shaped for cases* (I) *and* (II).

(1) *If $I - T$ is demi-closed, then T has a fixed point.*

(2) *If X satisfies Opial's condition, then T has a fixed point.*

(3) *If K is compact, then T has a fixed point.*

PROOF. Since T satisfies all the requirements of Theorem 1, there exist sequences $\{x_n\}$ and $\{y_n\}$ in K such that $y_n \in Tx_n$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$. Since K is weakly compact, we may assume that $x_n \to x$ weakly to some $x \in K$. Since $x_n - y_n \to 0$ strongly, $x_n - y_n \in (I - T)x_n$, and $I - T$ is demi-closed, we conclude that $0 \in (I - T)x$ and hence $x \in Tx$. This proves (1) and (2). Case (3) is clear.

Note that Zhang [14, Theorem 3.8 and Corollary 3.10] obtained Theorem 2(II), which extends earlier results of Lami Dozo [8], Itoh and Takahashi [6], Yanagi [13], and Markin [9].

From Theorem 2(1), we obtain the following:

THEOREM 3. *Let X be a uniformly convex Banach space, D a subset of X, and $f : D \to X$ a nonexpansive map. Let K be a nonempty closed (KR)-bounded subset of X such that $\overline{\text{co}}\, K \subset D$. Suppose that one of the following holds.*

(i) *K is star-shaped and $f(\text{Bd}\, K) \subset K$.*

(ii) *K is star-shaped and $fx \in \overline{I}_K(x)$ for all $x \in \text{Bd}\, K$.*

(iii) *$0 \in \text{Int}\, K$ and $fx \neq \lambda x$ for all $x \in \text{Bd}\, K$ and $\lambda \nabla 1$.*

Then f has a fixed point in $\overline{\text{co}}\, K$.

PROOF. By Theorem 1, as in the proof of [12, Theorem 3], there is a weakly compact subset L of $\overline{\text{co}}\, K$ and $I - f$ is demi-closed on L. See Browder [2]. Therefore, Theorem 3 follows from Theorem 2(1).

Theorem 3 was due to Park [12, Theorem 3]. Note that Gulevich [5, Theorem 1] obtained Theorem 3(i) for a bounded set K of a Hilbert space X without assuming the star-shapedness.

References

[1] N. A. Assad and W. A. Kirk, Fixed point theorems for set-valued mappings of contractive type, *Pacific J. Math.*, **43** (1972), 553–562.

[2] F. E. Browder, Semicontractions and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.*, **74** (1968), 660–665.

[3] A. Canetti, G. Marino, and P. Pietramala, Fixed point theorems for multivalued mappings in Banach spaces, *Nonlinear Anal. TMA*, **17** (1991), 11–20.

[4] K. Deimling, Fixed points of weakly inward multis, *Nonlinear Anal. TMA*, **10** (1986), 1261–1262.

[5] N. M. Gulevich, Existence of fixed points of nonexpansive mappings satisfying the Rothe condition, *J. Soviet Math.*, **26** (1984), 1607–1611.

[6] S. Itoh and W. Takahashi, Singlevalued mappings, multivalued mappings and fixed point theorems, *J. Math. Anal. Appl.*, **59** (1977), 154–521.

[7] W. A. Kirk and W. O. Ray, Fixed-point theorems for mappings defined on unbounded sets in Banach spaces, *Studia Math.*, **64** (1979), 127–138.

[8] E. Lami Dozo, Multivalued non-expansive mappings and Opial's condition, *Proc. Amer. Math. Soc.*, **38** (1973), 286–292.

[9] J. T. Markin, A fixed point theorem for set-valued mappings, *Bull. Amer. Math. Soc.*, **74** (1968), 639–640.

[10] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597.

[11] Sehie Park, Fixed point theory of multifunctions in topological vector spaces. II, *J. Korean Math. Soc.*, **30** (1993), 413–431.

[12] Sehie Park, On a problem of Gulevich on nonexpansive maps in uniformly convex Banach spaces, to appear.

[13] K. Yanagi, On some fixed point theorems for multivalued mappings, *Pacific J. Math.*, **87** (1980), 233–240.

[14] S. Zhang, Star-shaped sets and fixed points of multivalued mappings, *Math. Japonica*, **36** (1991), 327–334.

(Received May 30, 1994)

DEPARTMENT OF MATHEMATICS
SEOUL NATIONAL UNIVERSITY
SEOUL 151-742
KOREA