REMARKS ON SET-VALUED GENERALIZATIONS OF BEST APPROXIMATION THEOREMS

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1. INTRODUCTION

In our previous work [P4], most of known generalizations of the Fan or Prolla type best approximation theorems for single-valued maps are improved, extended, and unified. In the present paper, we mainly consider set-valued generalizations of best approximation theorems.

Usually such results are obtained for Kakutani maps; that is, upper semicontinuous multifunctions with compact convex values, defined on a convex subset of a locally convex Hausdorff topological vector space. However, in this paper, we deal with much larger class of multifunctions containing acyclic maps which are upper semicontinuous multifunctions with compact acyclic values, defined on a convex subset of a topological vector space having sufficiently many linear functionals. Our arguments are based on new types of fixed point theorems recently due to the author [P1,2].

Consequently, earlier works of Fan [F], Reich [R1,2], Prolla [Pr], Sehgal and Singh [SS], Sehgal, Singh, and Gastl [SSG], Carbone and Conti [CC], Ding and Tan [DT], Park, Singh, and Watson [PSW], Park [P3], and others are properly extended and improved.

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2. PRELIMINARIES

A *multifunction* or *set-valued map* (simply, *map*) $F : X \to 2^Y$ is a function with nonempty set-values $Fx \subset Y$ for each $x \in X$. The set $\{(x, y) : y \in Fx\}$ is called either the *graph* of F or, simply, F. So $(x, y) \in F$ if and only if $y \in Fx$. For any $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. For any $B \subset Y$, let $F^{-1}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$. If B is a singleton $\{y\}$ in Y, then $F^{-1}(B)$ is called a *fiber* denoted $F^{-1}y$.

For topological spaces X and Y, a map $F : X \to 2^Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-1}(B)$ is closed in X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^{-1}(B)$ is open in X; *continuous* if F is both u.s.c. and l.s.c.; and *compact* if $F(X)$ is contained in a compact subset of Y. A set $K \subset X$ is said to be *σ-compact* if K is a countable union of compact sets. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A *convex space* C is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*.

Given a class $\mathbb{L}$ of maps, $\mathbb{L}(X, Y)$ denotes the set of all maps $F : X \to 2^Y$ belonging to $\mathbb{L}$, and $\mathbb{L}_c$ the set of all finite composites of maps in $\mathbb{L}$.

A class $\mathfrak{A}$ of maps is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Aronszajn maps $\mathbb{M}$ (with R_δ values) [Gr], the O'Neill maps $\mathbb{N}$ (with values consisting of one or m acyclic components, where m is fixed) [Gr], the approachable maps $\mathbb{A}$ in topological vector spaces [BD 1-3], admissible maps in the sense of Górniewicz [G], permissible maps of Dzedzej [D], and others. Moreover, we define

$F \in \mathfrak{A}_c^\sigma(X,Y) \Longleftrightarrow$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(X,Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \Longleftrightarrow$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(X,Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [L] and $\mathbb{V}_c^\sigma$ due to Park, Singh, and Watson [PSW]. Note that $\mathbb{K}_c^\sigma$ contains classes $\mathbb{K}$, $\mathbb{R}$, and $\mathbb{T}$ in [L].

In this paper, *we assume that* $\mathbb{V} \subset \mathfrak{A}$; that is, *a class* $\mathfrak{A}_c^\kappa$ *always contains* $\mathbb{V}_c^\kappa$.

Let $E = (E, \tau)$ be a topological vector approximations to $y \in E$ from C by $Q_p(y) = \{x \in C : p(y-x) = d_p(y,C)\}$. The multifunction $Q_p : E \to 2^C$ thus defined is called the *metric projection* onto C if $Q_p(y) \neq \emptyset$ for each $y \in E$; that is, C is proximinal (with respect to p). The set C is said to be *approximatively compact* (with respect to p) if for each $y \in E$, every net $\{x_\alpha : \alpha \in \Lambda\} \subset C$ such that $p(y - x_\alpha) \to d_p(y,C)$ has a subnet that converges to an element of C.

The following is well-known [R1]:

Lemma 3. *Let C be a nonempty convex subset of a Hausdorff topological vector space E and $p \in S(E)$. If C is approximatively p-compact, then $Q_p \in \mathbb{K}(E,C)$.*

Note that every compact subset is approximatively compact and that every closed convex subset of a uniformly convex Banach space is approximatively norm-compact.

In (E,τ), let Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, respectively, with respect to τ.

The *inward* and *outward sets* of $X \subset E$ at $x \in E$, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = \{x + r(u-x) : u \in X,\ r > 0\},$$
$$O_X(x) = \{x + r(u-x) : u \in X,\ r < 0\}.$$

Let E be a vector space, p a seminorm on E, and C a nonempty convex subset of E. A function $g : C \to E$ is said to be

(i) *almost p-affine* if

$$p(g(rx + (1-r)y) - u) \leq rp(gx - u) + (1-r)p(gy - u);$$

(ii) *almost p-quasiconvex* if

$$p(g(rx + (1-r)y) - u) \leq \max\{p(gx - u), p(gy - u)\},$$

for $x, y \in C$, $u \in E$, and $r \in (0,1)$.

For a topological space X, a function $f : X \to \mathbb{R}$ is said to be *lower semicontinuous* (l.s.c.) if $\{x \in X : fx > r\}$ is open for each $r \in \mathbb{R}$.

The following is well known:

Lemma 4. *Let X and Y be topological spaces, $h : X \times Y \to \mathbb{R}$ l.s.c., and $F : X \to 2^Y$ a compact-valued u.s.c. multifunction. Then $x \mapsto \inf\{h(x,y) : y \in Fx\}$ is l.s.c. on X.*

3. MAIN RESULTS

The following is a main result in this paper:

Theorem 1. *Let (C, σ) be a convex space, $E = (E, \tau)$ a Hausdorff topological vector space containing C as a subset, $F : (C, \sigma) \to 2^E$, $p \in S(E, w)$, $Q_p : E \to 2^{(C,\tau)}$ the metric projection, and $g \in \mathbb{C}((C,\tau), E)$ such that $C \subset g(C)$. Suppose that either*

(I) E^* separates points of E, (C, σ) is compact, $g^{-1}Q_pF \in \mathfrak{A}_c^\kappa((C,\sigma),(C,\tau))$, and for each $q \in S(E, w)$, $(x, y) \mapsto q(x - y)$ is continuous on $(x, y) \in (C, \sigma) \times E$; or

(II) E is locally convex, $\sigma = \tau$ on C, and $g^{-1}Q_pF \in \mathfrak{A}_c^\sigma((C,\tau),(C,\tau))$ is compact.

Then there exists an $(x_0, y_0) \in F$ *such that*

$$p(gx_0 - y_0) = d_p(y_0, C).$$

Moreover, if $gx_0 \in C$*, then we have*

$$p(gx_0 - y_0) = d_p(y_0, \overline{I}_C(gx_0)).$$

In this case, $gx_0 \in \operatorname{Bd} C$ *and* $y_0 \notin \overline{I}_C(gx_0)$ *whenever* $p(gx_0 - y_0) > 0$.

Proof. Under the assumptions (I) or (II), by Lemmas 2 or 1, respectively, $g^{-1}Q_pF$ has a fixed point $x_0 \in (g^{-1}Q_pF)x_0$; that is, there exists an $(x_0, y_0) \in F$ such that $gx_0 \in Q_p(y_0)$; or equivalently

$$p(gx_0 - y_0) = d_p(y_0, C).$$

If $gx_0 \in C$, then for $z \in I_C(gx_0)\backslash C$, there exist $u \in C$ and $r > 1$ such that $z = gx_0 + r(u - gx_0)$. Suppose that $p(gx_0 - y_0) > p(z - y_0)$. Since

$$u = \frac{1}{r}z + (1 - \frac{1}{r})gx_0 \in C,$$

we have

$$p(u - y_0) \leq \frac{1}{r}p(z - y_0) + (1 - \frac{1}{r})p(gx_0 - y_0) < p(gx_0 - y_0),$$

a contradiction. Therefore, $p(gx_0 - y_0) \leq p(z - y_0)$ for all $z \in I_C(gx_0)$ and hence, for all $z \in \overline{I}_C(gx_0)$. Since $gx_0 \in \overline{I}_C(gx_0)$, we have

$$p(gx_0 - y_0) = d_p(y_0, \overline{I}_C(gx_0)).$$

Suppose that $p(gx_0 - y_0) > 0$. Then clearly $y_0 \notin \overline{I}_C(gx_0)$. Suppose that $gx_0 \in \operatorname{Int} C$. Then $\overline{I}_C(gx_0) = E$. Since $y_0 \in E$, we have $d_p(y_0, \overline{I}_C(gx_0)) = 0$, which is a contradiction. This completes our proof.

Remark. Since σ and τ on C may be different topologies in Case (I), we can consider various situations. In fact, in Case (I), σ and τ are related only by

(*) for each $q \in S(E, w), (x, y) \mapsto q(x - y)$ is continuous on $(C, \sigma) \times (E, \tau)$.

Therefore, it is sufficient to assume that

(1) as a convex space, (C, σ) has a topology σ finer than the relative one with respect to (E, w); and

(2) τ is any topology finer than w.

Corollary 1.1. *Under the hypothesis of Theorem 1, if $g(C) = C$, then the following equivalent statements hold:*

(i) *There exists an $(x_0, y_0) \in F$ such that*

$$p(gx_0 - y_0) = d_p(y_0, \overline{I}_C(gx_0)).$$

(ii) *If $H : C \to 2^E$ is a multifunction such that for any $(x, y) \in F$ with $gx \notin Hx$, there exists a $z \in \overline{I}_C(gx)$ satisfying*

$$p(gx - y) > p(z - y),$$

then $gx_0 \in Hx_0$ for some $x_0 \in C$.

(iii) *If $H : C \to 2^E$ is a multifunction such that $Hx \subset \overline{I}_C(gx)$ for all $x \in C$ and that, for each $(x, y) \in F$ with $gx \notin Hx$, there exists a $z \in Hx$ satisfying*

$$p(gx - y) > p(z - y),$$

then $gx_0 \in Hx_0$ for some $x_0 \in C$.

Proof. (i) $\Longrightarrow$ (ii) For the $(x_0, y_0) \in F$ in (i), suppose that $gx_0 \notin Hx_0$. Then, by assumption, for any $y \in Fx_0$ there exists a $z \in \overline{I}_C(gx_0)$ satisfying

$$p(gx_0 - y) > p(z - y).$$

This contradicts (i).

(ii) $\Longrightarrow$ (iii) Note that $Hx \subset \overline{I}_C(gx)$. Therefore, for each $(x,y) \in F$ with $gx \notin Hx$, $z \in Hx$ satisfies the inequality in (ii).

(iii) $\Longrightarrow$ (i) Suppose that, for any $(x,y) \in F$, there exists a $z \in \overline{I}_C(gx_0)$ such that

$$p(gx - y) > p(z - y).$$

Let $H : C \to 2^E$ be defined by $Hx = \{z \in \overline{I}_C(gx) : p(gx - y) > p(z - y)$ for some $y \in Fx\}$ for $x \in C$. Then $gx \notin Hx$ for all $x \in C$ by definition. However, for each $(x,y) \in F$, there exists a $z \in Hx$ satisfying the inequality in (iii). This contradicts (iii). Therefore, we should have (i). This completes our proof.

Particular Forms.

1. If C is a compact convex subset of a locally convex Hausdorff topological vector space E, $g = 1_C$, and $F = H \in \mathbb{K}(C,E)$, then Corollary 1.1(ii) reduces to Reich [R2, Theorems 1 and 2].

2. Some variants of Corollary 1.1 are obtained by Sehgal, Singh, and Gastle [SSG, Theorem 1 and Corollaries 1 and 3] for a continuous multifunction $F \in \mathbb{K}(C,E)$.

The following is basic for various coincidence or fixed point results, especially, for normed vector spaces.

Corollary 1.2. *Under the hypothesis of Theorem 1, if $g(C) = C$, then there exists an $(x_0, y_0) \in F$ such that $d_p(gx_0, Fx_0) = 0$ whenever one of the following conditions holds:*

(i) *For each $(x,y) \in F$, $p(gx - y) > 0$ implies $p(gx - y) > d_p(y, \overline{I}_C(gx))$.*

(ii) *For each $(x,y) \in F$, there exists a number λ (real or complex, depending on whether E is real or complex) such that*

$$|\lambda| < 1 \quad \textit{and} \quad \lambda gx + (1 - \lambda)y \in \overline{I}_C(gx).$$

(iii) *For each $x \in X$, we have $Fx \subset \overline{I}_C(gx)$.*

Proof. (i) For the $(x_0, y_0) \in F$ in the conclusion of Theorem 1, suppose that $d_p(gx_0, Fx_0) > 0$. Then, for $y_0 \in Fx_0$, we have $p(gx_0 - y_0) > d_p(y_0, \overline{I}_C(gx_0))$, which contradicts the conclusion of Theorem 1.

(ii) Suppose that $p(gx - y) > 0$ for $(x, y) \in F$. By putting $z = \lambda gx + (1 - \lambda)y \in \overline{I}_C(gx)$, we have

$$p(z - y) \leq |\lambda| p(gx - y) < p(gx - y)$$

and hence

$$d_p(y, \overline{I}_C(gx)) < p(gx - y).$$

Therefore, (ii) implies (i).

(iii) Put $\lambda = 0$ in (ii).

Particular Form. For $\sigma = \tau$, $g = 1_C$ and $\mathbb{V}_c^\sigma$ instead of $\mathfrak{A}_c^\sigma$, a normed vector space version of Corollary 1.2 is due to Park, Singh, and Watson [PSW, Corollary 1].

Recall that a map $g : X \to Y$ is *proper* if $g^{-1}(K)$ is compact whenever K is compact in Y.

The following is an example satisfying the hypothesis of Theorem 1:

Corollary 1.3. *Let C be a nonempty convex subset of a normed vector space E and $Q : E \to 2^C$ the metric projection satisfying*

(1) *$Qx \neq \emptyset$ for each $x \in E$; and*

(2) *Q maps compact subsets of E onto compact subsets of C.*

Let $g : C \to C$ be a continuous, proper, almost quasiconvex surjection and $f : C \to E$ a compact continuous map.

Then there exists an $x_0 \in C$ such that

$$\|gx_0 - fx_0\| = d(fx_0, \overline{I}_C(gx_0)) = d(fx_0, C).$$

Proof. As in the proof of Sehgal and Singh [SS, Theorem 3], we can show that $g^{-1}Qf \in \mathbb{K}(C, C)$. Since f is compact, Q satisfies (2), and g is proper, we know that $g^{-1}Qf$ is compact. Therefore, by Theorem 1(II), we have the conclusion.

Particular Forms.

1. If g is almost affine instead of almost quasiconvex, then Corollary 1.3 reduces to Sehgal and Singh [SS, Theorem 3].

2. If C is approximatively compact, then Q satisfies conditions (1) and (2) of Corollary 1.3. See Lemma 3.

By putting $\sigma = \tau$ in Theorem 1, we obtain the following useful result:

Theorem 2. *Let C be a convex subset of a Hausdorff topological vector space E, $p \in S(E)$ and $g \in \mathbb{C}(C, C)$ with acyclic fibers. Suppose that either*

(I) *E^* separates points of E, C is compact, and $F \in \mathfrak{A}_c^\kappa(C, E)$; or*

(II) *E is locally convex, C is approximatively p-compact, $F \in \mathfrak{A}_c^\sigma(C, E)$ is compact, and g is proper.*

Then there exists an $(x_0, y_0) \in F$ such that

$$p(gx_0 - y_0) = d_p(y_0, \overline{I}_C(gx_0)).$$

Moreover, $gx_0 \in \operatorname{Bd} C$ and $y_0 \notin \overline{I}_C(gx_0)$ whenever $p(gx_0 - y_0) > 0$.

Proof. Since g has acyclic fibers, we have $C = g(C)$. Consider the metric projection $Q_p : E \to 2^C$. Then by Lemma 3, $Q_p \in \mathbb{K}(E, C) \subset \mathfrak{A}(E, C)$.

(I) Note that $g^{-1} \in \mathbb{V}(C, C) \subset \mathfrak{A}(C, C)$. Since $\mathfrak{A}_c$ is closed under composition, we have $g^{-1}Q_pF \in \mathfrak{A}_c^\kappa(C, C)$.

(II) Since F is compact and Q_p is u.s.c. and compact-valued, we know Q_pF is compact. Let $K = \overline{(Q_pF)(C)} \subset C$. Then $g^{-1}|_K$ has the closed graph with Hausdorff compact range since g is proper. Therefore $g^{-1}|_K \in \mathbb{V}(K, C) \subset \mathfrak{A}(K, C)$. Therefore, $g^{-1}Q_pF \in \mathfrak{A}_c^\sigma(C, C)$ is compact.

In any case, by Theorem 1, the conclusion follows.

Particular Forms.

1. If C itself is compact, $g = 1_C$, and $F = f \in \mathbb{C}(C, E)$, then Theorem 2 for a normed vector space is just Fan [F, Theorem 2], which is the origin of Theorem 2.

2. For $g = 1_C$ and $F = f \in \mathbb{C}(C, E)$, Theorem 2(II) reduces to Reich [R1, Corollary 2.2].

3. If C itself is compact in (II), then $g \in \mathbb{C}(C, C)$ is clearly proper. In this case, Theorem 2 for a normed vector space and for $F = f \in \mathbb{C}(C, E)$ improves Carbone and Conti [CC, Corollary 1]. Moreover, for an almost affine map g, Theorem 2 for a normed vector space reduces to Prolla [Pr, Theorem].

4. If p is a norm and $F = f$, Theorem 2(II) improves Carbone and Conti [CC, Theorem and Corollary 2].

5. For $g = 1_C$ and $\mathbb{V}_c^\sigma$ instead of $\mathfrak{A}_c^\sigma$, Theorem 2(II) reduces to Park, Singh, and Watson [PSW, Theorem 3].

6. For $g = 1_C$, Theorem 2(II) reduces to Park [P3, Theorem 5].

Under the hypothesis of Theorem 2(I), we obtain a very general coincidence theorem.

Theorem 3. *Let C be a compact convex subset of a topological vector space E, $F \in \mathfrak{A}_c^\kappa(C, E)$, $g \in \mathbb{C}(C, C)$ with acyclic fibers. If E^* separates points of E, then there exists an $x_0 \in C$ such that $gx_0 \in Fx_0$ whenever one of the conditions* (i)–(iii) *in Corollary 1.2 holds for each $p \in S(E, w)$.*

Proof. Since (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) as shown in the proof of Corollary 1.2, it suffices to assume (i). Note that the hypothesis of Theorem 1(I) is satisfied for each $p \in S(E, w)$ as shown in the proof of Theorem 2. Moreover, since C is compact, we may regard $F \in \mathfrak{A}_c(C, E)$. Therefore, by Corollary 1.2, for each $p \in S(E, w)$ we have

$$F_p = \{x \in C : d_p(gx, Fx) = 0\} \neq \emptyset.$$

Since F is u.s.c. and compact-valued, by Lemma 4, F_p is a nonempty closed subset of C. Further, $\{F_p : p \in S(E, w)\}$ has the finite intersection property. In fact, for each $\{p_1, p_2, \cdots, p_n\} \subset S(E,w)$ we have $p = \sum_{i=1}^n p_i \in S(E,w)$ and $F_p \subset \bigcap_{i=1}^n F_{p_i}$. Since C is compact, we have an $u \in \bigcap\{F_p : p \in S(E,w)\} \neq \emptyset$.

Now, we claim that $gu \in Fu$. Suppose that $gu \notin Fu$. Then the origin 0 does not belong to the compact set $K = gu - Fu$. For each $z \in K$ there exists a $p_z \in S(E,w)$ such that $p_z(z) > 0$. Since p_z is continuous on E, there exists an open neighborhood U_z of z such that $p_z(y) > 0$ for every $y \in U_z$. Let $\{U_{z_1}, \cdots, U_{z_k}\}$ be a finite subcover of the cover $\{U_z\}_{z\in K}$ of K and $p_u = \sum_{i=1}^k p_{z_i} \in S(E,w)$. Since $p_u|_K$ is continuous, it attains its infimum on K. Since the minimum can not be zero, we have $d_{p_u}(gu, Fu) > 0$. This contradicts $u \in \bigcap\{F_p : p \in S(E,w)\} \neq \emptyset$. This completes our proof.

Particular Forms.

1. For a locally convex Hausdorff topological vector space E, $F \in \mathbb{K}(C,E)$, and $g = 1_C$, Theorem 3 for (i) reduces to Reich [R1, Theorem 3.1], [R2, Theorem 2], and improves Sehgal, Singh, and Gastl [SSG, Corollary 1].

2. If E has the weak topology, $F \in \mathbb{K}(C,E)$, and $g = 1_C$, then Theorem 3 for (i) improves Ding and Tan [DT, Corollary 1].

3. For a locally convex Hausdorff topological vector space E with the weak topology and $F \in \mathbb{K}(C,E)$, Theorem 3 reduces to Ding and Tan [DT, Theorems 4–6].

4. For a normed vector space E and $F = f \in \mathbb{C}(C,E)$, Theorem 3 extends Sessa and Singh [SeS, Theorem 4].

5. For $g = 1_C$, Theorem 3 reduces to Park [P3, Theorem 4].

Recall that a reflexive Banach space E has the *Oshman property* if the metric projection Q on every closed convex subset belongs to $\mathbb{K}(E,C)$.

Theorem 4. *Let C be a closed convex subset of a Banach space E with the Oshman property, $F \in \mathfrak{A}_c^\kappa(C,E)$ a compact map, $g \in \mathbb{C}(C,C)$ a proper map with acyclic fibers, Then there exists an $(x_0, y_0) \in F$ such that*

$$\|gx_0 - y_0\| = d(y_0, C) = d(y_0, \overline{I}_C(gx_0)).$$

Proof. Just follow the proof of Theorem 2(I).

Particular Forms.

1. For $g = 1_C$ and $F = f \in \mathbb{C}(C, E)$, Theorem 4 is due to Reich [R2, Proposition 2.3].

2. For $g = 1_C$ and $\mathbb{K}_c^\sigma$ instead of $\mathfrak{A}_c^\kappa$, Theorem 4 reduces to Park, Singh, and Watson [PSW, Theorem 4].

3. For $g = 1_X$, Theorem 4 reduces to Park [P3, Theorem 6].

Corollary 4.1. *Under the hypothesis of Theorem 4, there exists an $x_0 \in C$ such that $gx_0 \in Fx_0$ if one of the following conditions holds:*

(i) *For each $x \in C$ with $d(gx, Fx) > 0$ and each $y \in Fx$, there exists a $z \in \overline{I}_C(gx)$ such that*

$$\|gx - y\| > \|z - y\|.$$

(ii) *For each $(x, y) \in F$, there exists a number λ (as in Corollary 1.2) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda gx + (1 - \lambda)y \in \overline{I}_C(gx).$$

(iii) *For each $x \in C$, $Fx \subset \overline{I}_C(gx)$.*

Particular Forms.

1. For $g = 1_C$ and $\mathbb{K}$ instead of $\mathfrak{A}_c^\kappa$, Corollary 4.1 reduces to Reich [R1, Proposition 3.2 and Theorem 3.3], [R2, Theorems 6 and 7].

2. For $g = 1_C$ and $\mathbb{V}_c^\sigma$ instead of $\mathfrak{A}_c^\kappa$, Corollary 4.1 reduces to Park, Singh, and Watson [PSW, Corollary 2].

3. For $g = 1_C$, Corollary 4.1 reduces to Park [P3, Corollary 4].

Final Remark. In Theorems 2 and 4, if $F \in \mathbb{K}(C, E)$ and F is l.s.c. (hence, F is continuous), then some authors obtained the following conclusion:

(A) There exists an $x_0 \in C$ such that

$$d_p(gx_0, Fx_0) \leq d_p(z, Fx_0) \quad \text{for all} \quad z \in \overline{I}_C(gx_0).$$

Note that (A) implies the following conclusion of Theorems 2 and 4:

(B) There exists an $(x_0, y_0) \in F$ such that

$$p(gx_0 - y_0) \leq p(z - y_0) \quad \text{for all} \quad z \in \overline{I}_C(gx_0).$$

In fact, since Fx_0 is compact there exists an $y_0 \in Fx_0$ such that $d_p(gx_0, Fx_0) = p(gx_0 - y_0)$ and hence

$$p(gx_0 - y_0) \leq d_p(z, Fx_0) \leq p(z - y_0) \text{ for all } z \in \overline{I}_C(gx_0).$$

Note that there are examples of $F \in \mathbb{K}(C, E)$ which does not satisfy (A). Therefore, in order to ensure (A), the lower semicontinuity of F is not dispensable.

However, for $F \in \mathbb{V}(C, E)$, (A) can not be true even if F is continuous and $g = 1_C$.

Example. Let $C = [-1, 1] \times \{0\} \subset \mathbb{R}^2 = E$, $g = 1_C$, and

$$Fx = [(-2, 0), (0, 2)] \cup [(0, 2), (2, 0)]$$

for $x \in C$, where $[P, Q]$ stands for the line segments joining points $P, Q \in \mathbb{R}^2$. Then $F : C \to E$ is constant map and hence continuous. Note that, obviously, there is no $x_0 \in C$ satisfying (A). However, $x_0 = (1, 0)$ and $y_0 = (3/2, 1/2)$ satisfies (B).

Therefore, for acyclic maps or maps in more general classes the conclusion (B) seems to be more natural than (A).

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