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Coincidence Theorems for Admissible Multifunctions on Generalized Convex Spaces*

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We defined admissible classes of maps which are general enough to include composites of maps appearing in nonlinear analysis or algebraic topology, and generalized convex spaces which are generalizations of many general convexity structures. In this paper we obtain a coincidence theorem for admissible maps defined on generalized convex spaces. Our new result is applied to obtain an abstract variational inequality, a KKM type theorem, and fixed point theorems.

1. INTRODUCTION

Recently the first author introduced admissible multifunctions (maps) and generalized convex (or G-convex) spaces which are adequate to establish theories on fixed points, coincidence points, KKM maps, variational inequalities, best approximations, and many others. For details, see [74, 75, 77, 78, 80].

Our admissible classes of maps are very general enough to include composites of important maps which appear in nonlinear analysis or algebraic topology. And our concept of generalized convex spaces is a generalization of many general convexities which were developed in connection mainly with the fixed point theory and the KKM theory. See [80].

In this paper we obtain a coincidence theorem for admissible maps defined on G-convex spaces. This new result is applied to obtain an abstract variational inequality, a KKM type theorem, and fixed point theorems. Each of our results includes a large number of known theorems as particular cases. See References.

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The origin of our coincidence theorem is known as the Fan–Browder fixed point theorem due to Fan [27] and Browder [16-17]. In fact, using his own generalization of the classical KKM theorem [51], Ky Fan [27] established an elementary but very basic "geometrical" lemma for multifunctions. Later Browder [16] restated this result in the more convenient form of a fixed point theorem by means of the Brouwer fixed point theorem and the partition of unity argument. Since then, there have appeared numerous generalizations and applications in various fields such as fixed point theory, minimax theory, and variational inequalities. Many of these results are unified and improved in this paper.

2. PRELIMINARIES

A *multifunction* (or *map*) $F: X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-}y = \{x \in X: y \in Fx\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{Fx: x \in A\}$. For any $B \subset Y$, the (*lower*) *inverse* and (*upper*) *inverse* of B under F are defined by

$$F^{-}(B) = \{x \in X: Fx \cap B \neq \varnothing\} \text{ and } F^{+}(B) = \{x \in X: Fx \subset B\},$$

resp. The (*lower*) *inverse* of $F: X \multimap Y$ is the map $F^{-}: Y \multimap X$ defined by $x \in F^{-}y$ if and only if $y \in Fx$. Given two maps $F: X \multimap Y$ and $G: Y \multimap Z$, the *composite* $GF: X \multimap Z$ is defined by $(GF)x = G(Fx)$ for $x \in X$.

For topological spaces X and Y, a map $F: X \multimap Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-}(B)$ is closed in X. A map $F: X \multimap Y$ is *compact* provided $F(X)$ is contained in a compact subset of Y.

Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact.

Let $\overline{}$ denote the closure.

Let $\mathscr{V}$ be the fundamental system of neighborhoods of the origin 0 in a topological vector space (simply, t.v.s.) E. In E, a convex hull of its finite subset will be called a *polytope*.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F: X - 1 \circ Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class of $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values) [50], the Aronszajn maps $\mathbb{M}$ (with R_δ values) [33], the acyclic maps $\mathbb{V}$ (with acyclic values) [25], the O'Neill maps $\mathbb{N}$ (with values consisting of one or m acyclic components, where m is fixed) [33], the approachable maps $\mathbb{A}$ in uniform spaces [8], admissible maps in the sense of Górniewicz [31], permissible maps of Dzedzej [24], and others. For details, see [80].

A class $\mathfrak{A}_c^\sigma$ is defined as follows:

$F \in \mathfrak{A}_c^\sigma(X,Y) \Leftrightarrow$ for any σ-compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}_x \subset Fx$ for each $x \in K$.

$\mathfrak{A}_c^\sigma$ is due to Park [77]. Further, a class $\mathfrak{A}_c^\kappa$ is defined as follows:

$F \in \mathfrak{A}_c^\kappa(X,Y) \Leftrightarrow$ for any compact subset K of X, there exists an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

$\mathfrak{A}_c^\kappa$ is due to Park [74, 77, 78] and will be called *admissible*.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [58] and $\mathbb{V}_c^\sigma$ due to Park *et al.* [81]. Note that $\mathbb{K}_c^\sigma$ contains $\mathbb{K}$, Fan–Browder type maps [16, 27], and $\mathbb{T}$ in [58].

For a nonempty set D, let $\langle D \rangle$ denote the set of all nonempty finite subsets of D. For a set A, let $|A|$ denote the cardinality of A. Let Δ_n denote the standard n-simplex, that is,

$$\Delta_n = \left\{ u \in \mathbb{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u) e_i,\ \lambda_i(u) \geq 0,\ \sum_{i=1}^{n+1} \lambda_i(u) = 1 \right\},$$

where e_i is the ith unit vector in $\mathbb{R}^{n+1}$.

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* [77] if convex hulls of any nonempty finite subsets of D are contained in X and X has a topology that induces the Euclidean topology on such convex hulls. A subset A of X is said to be *D-convex* if, for each $N \in \langle D \rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$, where co denotes the convex hull. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [55].

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ [80] consists of a topological space X, a nonempty subset D of X, and a map $\Gamma: \langle D \rangle \multimap X$ with nonempty values such that

(1) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and

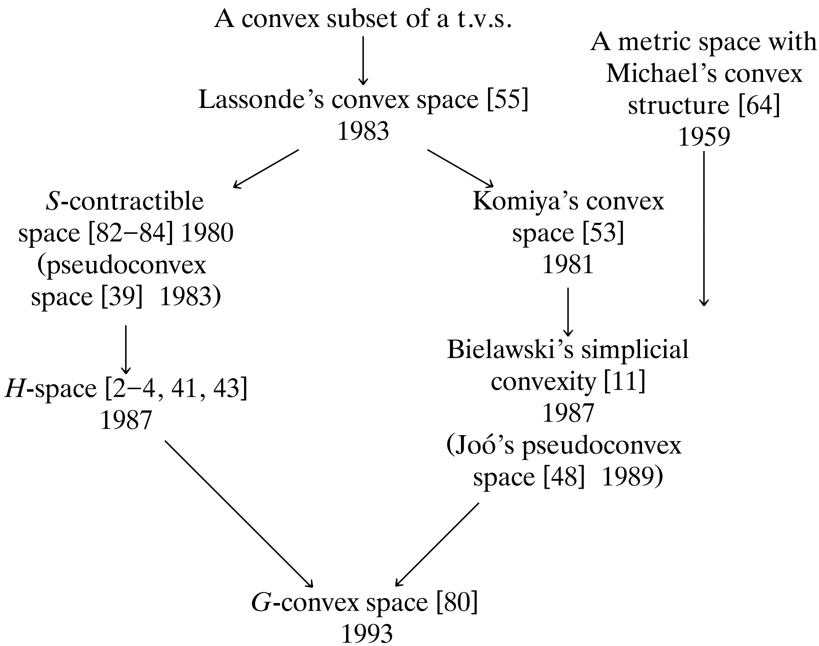
(2) for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A: \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D \rangle$. For an $(X, D; \Gamma)$, a subset C of X is said to be *G-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

Note that Γ_A does not need to contain A for $A \in \langle D \rangle$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$.

Any convex space (X, D) becomes a G-convex space $(X, D; \Gamma)$ by putting $\Gamma_A = \mathrm{co}\, A$.

The major particular forms of G-convex spaces can be adequately summarized by the following diagram. In the diagram, we may regard Horvath's pseudoconvex spaces as S-contractible spaces and Joó's pseudoconvex spaces as spaces with simplicial convexity, resp., for simplicity. For details, see [80].



3. MAIN RESULTS

We begin with the following coincidence theorem:

THEOREM 1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S: D \multimap Y$, $T: X \multimap Y$ maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1.1) *for each $x \in D$, Sx is compactly open in Y;*

(1.2) *for each $y \in F(X)$, $M \in \langle S^- y \rangle$ implies $\Gamma_M \subset T^- y$;*

(1.3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*

(1.4) *either*

(i) $Y \setminus K \subset S(M)$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact G-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \setminus K \subset S(L_N \cap D)$.

Then there exists an $\bar{x} \in X$ *such that* $F\bar{x} \cap T\bar{x} \neq \varnothing$.

Proof. Since $\overline{F(X)} \cap K$ is compact and covered by compactly open sets Sx by (1.1) and (1.3), there exists an $N \in \langle D \rangle$ such that $\overline{F(X)} \cap K \subset S(N)$.

Case (i). Since $Y \setminus K \subset S(M)$ for some $M \in \langle D \rangle$ by (i), we have $\overline{F(X)} \subset S(A)$, where $A = M \cup N = \{x_1, x_2, \ldots, x_{n+1}\} \in \langle D \rangle$. Then, there exist a $\phi_A \in \mathbb{C}(\Delta_n, X)$ such that $\phi_A(\Delta_n) \subset \Gamma_A$ and $\phi_A(\Delta_J) \subset \Gamma_J$ for each $J \in \langle A \rangle$, an $\tilde{F} \in \mathfrak{A}_c(\phi_A(\Delta_n), Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in \phi_A(\Delta_n)$, and $\{\lambda_i\}_{i=1}^{n+1}$ the partition of unity subordinated to the cover $\{Sx_i \cap \tilde{F}\phi_A(\Delta_n)\}_{i=1}^{n+1}$ of $\tilde{F}\phi_A(\Delta_n)$.

Define a continuous map $p: \tilde{F}\phi_A(\Delta_n) \to \Delta_n$ by

$$p(y) = \sum_{i=1}^{n+1} \lambda_i(y) e_i = \sum_{i \in N_y} \lambda_i(y) e_i \qquad \text{for } y \in \tilde{F}\phi_A(\Delta_n),$$

where $i \in N_y \Leftrightarrow \lambda_i(y) \neq 0 \Rightarrow y \in Sx_i \Leftrightarrow x_i \in S^- y$. By (1.2), we have $(\phi_A p)y \in \phi_A(\Delta_{N_y}) \subset \Gamma_{N_y} \subset T^- y$ for each $y \in \tilde{F}\phi_A(\Delta_n)$; that is, $y \in (T\phi_A p)y$.

Since $p\tilde{F}\phi_A \in \mathfrak{A}_c(\Delta_n, \Delta_n)$, $p\tilde{F}\phi_A$ has a fixed point $z \in \Delta_n$; that is, $z \in (p\tilde{F}\phi_A)z$. Put $\bar{x} = \phi_A(z)$. Since $p^- z \cap (\tilde{F}\phi_A)z = p^- z \cap \tilde{F}\bar{x} \neq \varnothing$, for any $y \in p^- z \cap \tilde{F}\bar{x}$, we have $y \in \tilde{F}\phi_A(\Delta_n)$, $(\phi_A p)y = \phi_A(z) = \tilde{x}$, and $y \in (T\phi_A p)y = T\bar{x}$. Therefore, $p^- z \cap \tilde{F}\bar{x} \subset T\bar{x}$ and hence $T\bar{x} \cap \tilde{F}\bar{x} \subset T\bar{x} \cap F\bar{x} \neq \varnothing$.

Case (ii). For an $N \in \langle D \rangle$ such that $\overline{F(X)} \cap K \subset S(N)$, consider the set L_N in (1.4).

We claim that $\tilde{F}(L_N) \subset S(L_N \cap D)$ for $\tilde{F} \in \mathfrak{A}_c(L_N, Y)$ satisfying $\tilde{F}x \subset Fx$ for each $x \in L_N$. In fact, note that

$$\tilde{F}(L_N) \cap K \subset F(X) \cap K \subset S(N) \subset S(L_N \cap D).$$

On the other hand, $\tilde{F}(L_N) \setminus K \subset F(L_N) \setminus K \subset S(L_N \cap D)$ by (1.4). Therefore, we have $\tilde{F}(L_N) \subset S(L_N \cap D)$.

Note that $\tilde{F}(L_N)$ is compact since it is the image of the compact set L_N under $\tilde{F}$. Therefore, $\tilde{F}(L_N) \subset S(A)$ for some $A = \{x_1, x_2, \ldots, x_{n+1}\} \in \langle L_N \cap D \rangle$.

For the remainder of the proof, we can just follow that of Case (i) and show that $T\bar{x} \cap \tilde{F}\bar{x} \subset T\bar{x} \cap F\bar{x} \neq \emptyset$ for some $\bar{x} \in L_N$. This completes our proof.

Remarks. 1. If X is a convex space with $\Gamma_A = \operatorname{co} A$, then (i) implies (ii). In fact we can choose $L_N = \operatorname{co}(M \cup N)$. However, in general, we cannot say (i) $\Rightarrow$ (ii) for G-convex spaces.

2. Note that the Hausdorffness of Y is necessary for the partition of unity argument in the proof. If F is single-valued we do not need to assume the Hausdorffness of Y.

3. Note that (1.2) generalizes the following:

(1.2)′ *for each* $x \in D$, $Sx \subset Tx$ *and* $T^{-}y$ *is* G-*convex for each* $y \in F(X)$, as in Park [77, Theorem 5] for convex spaces and [73, Theorem 3] for H-spaces.

4. If F is compact, then by putting $K = \overline{F(X)}$, condition (1.4) holds automatically.

Particular forms for compact admissible maps. 1. For convex spaces instead of G-convex spaces, Theorem 1 includes Browder [16, Theorem 1], Tarafdar and Husain [102, Theorem 1.1], Ben-El-Mechaiekh *et al.* [10, II, Théorème 3.1, 4.1, 4.2 and Corollaire 3.4], Simons [91, Theorem 4.3], Takahashi [97, Theorem 5], Browder [18, Theorem 4], Komiya [54, Theorem 1], Granas and Liu [35, Theorem 4.1], Lassonde [58, Theorem 4], Park *et al.* [81, Theorem 1], and Park [77, Theorem 2].

2. For other particular types of G-convex, Theorem 1 includes Komiya [53, Theorem 1], Bielawski [11, Propositions 4.9 and 4.12], Horvath [42, Corollaire 6 and 7; 41, I, Theorem 2′; 43, Corollary 4.2], and Park and Kim [79, Corollary 3.2].

Particular forms for non-compact admissible maps. 1. For convex spaces, Theorem 1 reduces to Park [77, Theorem 5], and for H-spaces it reduces to Park and Kim [79, Theorem 1].

2. For $\mathbb{V}$ instead of $\mathfrak{A}_c^{\kappa}$, Theorem 1 reduces to Park [70, Theorem 1], which includes earlier works of Browder [16–18], Tarafdar [98–101], Tarafdar and Husain [102], Ben-El-Mechaiekh *et al.* [9, 10], Yannelis and Prabhakar [104], Lassonde [55, 56], Ko and Tan [52], Simons [92, 93], Takahashi [97], Komiya [54], Mehta [62], Mehta and Tarafdar [63], Sessa [89], Jiang [45–47], McLinden [61], Granas and Liu [34, 35], Park [66–68], and Chang [19].

3. For an H-space $X = Y$ and $F = 1_X$, Theorem 1 contains Horvath [39, Théorème 4.1; 40, Théorème 2 and Lemme 1; 41, I, Theorem 2′; 43, Theorem 3.2], Ding and Tan [23, Theorems 10–12 and Corollaries 2–4], Ding *et al.* [22, Corollaries 3–5], Tarafdar [101, Theorem 2], Chen [21, Theorem 2], and Park [72, Theorem 6; 73, Theorem 4].

Among the numerous applications of Theorem 1, we give an abstract variational inequality:

THEOREM 2. *Let* $(X, D; \Gamma)$ *be a Hausdorff G-convex space,* $h: X \to [-\infty, \infty]$ *with* $h \not\equiv \infty$, $p: X \times X \to (-\infty, \infty]$, $q: D \times X \to (-\infty, \infty]$, $F \in \mathfrak{A}_c^\kappa(X, X)$, *and K a nonempty compact subset of X. Suppose that*

(2.1) $q(x, y) \le p(x, y)$ *for* $(x, y) \in D \times X$, *and* $p(x, y) + h(y) \le h(x)$ *for* $x \in X$ *and* $y \in Fx$;

(2.2) *for each* $x \in D$, $\{y \in X: q(x, y) + h(y) > h(x)\}$ *is compactly open;*

(2.3) *for each* $y \in F(X)$, $\{x \in X: p(x, y) + h(y) > h(x)\}$ *is G-convex; and*

(2.4) *either*

(i) $Y \setminus K \subset \bigcup_{x \in M}\{y \in X: q(x, y) + h(y) > h(x)\}$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact G-convex subset* L_N *of X containing N such that*

$$F(L_N) \setminus K \subset \bigcup_{x \in L_N \cap D} \{y \in X: q(x, y) + h(y) > h(x)\}.$$

Then there exists a solution $y_0 \in \overline{F(X)} \cap K$ *of the variational inequality*

$$q(x, y_0) + h(y_0) \le h(x) \qquad \text{for all } x \in D.$$

Moreover, the set of all solutions y_0 *is a compact subset of* $\overline{F(X)} \cap K$.

Proof. Define maps $S: D \multimap X$ and $T: X \multimap X$ by

$$Sx = \{y \in X: q(x, y) + h(y) > h(x)\} \qquad \text{for } x \in D,$$

and

$$Tx = \{y \in X: p(x, y) + h(y) > h(x)\} \qquad \text{for } x \in X.$$

Then (1.2) is satisfied, since $S^- y \subset T^- y$ for each $y \in X$ and $T^- y$ is G-convex. Suppose that there exists a $y_0 \in \overline{F(X)} \cap K$ such that $y_0 \notin S(D)$. Then the conclusion follows. Therefore we may assume that $\overline{F(X)} \cap K \subset S(D)$. Then all of the requirements of Theorem 1 are satisfied. Hence, there exists an $x_0 \in X$ such that $Fx_0 \cap Tx_0 \neq \emptyset$. Let $y_0 \in Fx_0 \cap Tx_0$. Then $y_0 \in Fx_0$ and

$$p(x_0, y_0) + h(y_0) > h(x_0),$$

which contradicts (2.1). Moreover, the set of all solutions y_0 is the intersection

$$\bigcap_{x \in D} \{y \in \overline{F(X)} \cap K: q(x, y) + h(y) \le h(x)\}$$

of compactly closed subsets of the compact set $\overline{F(X)} \cap K$. This completes our proof.

Remark. If $X = K$ itself is compact, then $y_0 \in Fx_0$ for some $x_0 \in X$. Even for $F = 1_X$, Theorem 2 is a basis of existence theorems of many results concerning variational inequalities. See [32, 37, 70].

Particular forms. For $F = 1_X$, there have appeared a lot of particular forms of Theorem 2. See Brézis *et al.* [13], Juberg and Karamardian [49], Mosco [65], Allen [1], Takahashi [96], Gwinner [37], Lassonde [55], Park [69], and Ben-El-Mechaiekh [6].

From Theorem 1 we obtain the following KKM theorem for G-convex spaces:

THEOREM 3. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G: D \multimap Y$ be a map such that*

(3.1) *for each $x \in D$, Gx is compactly closed in Y;*
(3.2) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*
(3.3) *there exists a nonempty compact subset K of Y such that either*
(i) *$\bigcap\{Gx: x \in M\} \subset K$ for some $M \in \langle D \rangle$; or*
(ii) *for each $N \in \langle D \rangle$, there exists a compact G-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{Gx: x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx: x \in D\} \neq \emptyset$.

Proof. Suppose the conclusion does not hold. Then $\overline{F(X)} \cap K \subset S(D)$, where $Sx = Y \setminus Gx$ for $x \in D$. Let $H: Y \multimap X$ and $T: X \multimap Y$ be defined by $Hy = \bigcup\{\Gamma_M: M \in \langle S^- y\rangle\}$ for $y \in Y$ and $Tx = H^- x$ for $x \in X$. Then all of the requirements of Theorem 1 are satisfied, and hence T and F have a coincidence point $x_0 \in X$; that is, $Tx_0 \cap Fx_0 \neq \emptyset$. For $y \in Tx_0 \cap Fx_0$, we have $x_0 \in T^- y = \bigcup\{\Gamma_M: M \in \langle S^- y\rangle\}$, and hence there exists a finite set M in $S^- y \subset D$ such that $x_0 \in \Gamma_M$. Since $M \in \langle S^- y\rangle$ implies $y \in Sx$ for all $x \in M$, we have $y \in Fx_0 \cap \bigcap\{Sx: x \in M\} \subset F(\Gamma_M) \cap \bigcap\{Sx: x \in M\}$; that is, $F(\Gamma_M) \not\subset G(M)$. This contradicts (3.2).

Remark. Condition (3.2) is equivalent to $\Gamma_N \subset F^+ G(N)$. A KKM type theorem for this case different from Theorem 3 can be found in [70, Theorem 4].

Particular forms. 1. The origin of Theorem 3 goes back to Sperner [95] and Knaster *et al.* [51] for $X = Y = K = \Delta_n$ an n-simplex, D its set of vertices, and $F = 1_X$.

2. For a convex space X, Theorem 3 reduces to Park [77, Theorem 7]. As Park noted in [70], a particular form [70, Theorem 3] of [77, Theorem 7] for $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$ includes earlier works of Fan [27–29], Lassonde [55],

Chang [19], and Park [67, 68]. Moreover, Park [76] showed that [70, Theorem 3] also extends a number of KKM type theorems due to Sehgal *et al.* [88], Lassonde [57], Shioji [90], Liu [59], Chang and Zhang [20], and Guillerme [36].

3. For an H-space X and $F = 1_X$, Theorem 3 generalizes Horvath [39, Théorème 3.1 and Corollaire 3; 41, I, Theorem 1 and Corollary 1], Bardaro and Ceppitelli [2, Theorem 1], Ding and Tan [23, Corollary 1 and Theorem 8], Ding *et al.* [22, Lemma 1], and Park [71, Theorems 1 and 4; 73, Theorems 1 and 3].

From Theorem 1, we obtain the following:

THEOREM 4. *Let (X, D) be a convex space, E a Hausdorff t.v.s. containing X as a subset, $F \in \mathfrak{A}_c^\kappa(X, E)$, and V a convex open neighborhood of the origin of E. Suppose that there exists a nonempty compact subset K of E such that*

(a) $\overline{F(X)} \cap K \subset D + V$; *and*

(b) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that $F(L_N) \setminus K \subset (L_N \cap D) + V$.*

Then F has a V-fixed point $x_V \in X$; that is, $Fx_V \cap (x_V + V) \neq \emptyset$.

Proof. Define $S: D \multimap E$ by $Sx = x + V$ for $x \in D$ and $T: X \multimap E$ by $Tx = x + V$ for $x \in X$. Then

(1.1) Sx is open for each $x \in D$;
(1.2) $\operatorname{co} S^- y = \operatorname{co}((y - V) \cap D) \subset T^- y = (y - V) \cap X$ for each $y \in E$;
(1.3) $\overline{F(X)} \cap K \subset S(D)$ by (a); and
(1.4) $F(L_N) \setminus K \subset S(L_N \cap D) = (L_N \cap D) + V$ by (b).

Therefore, by Theorem 1, F and T have a coincidence point $x_V \in X$; that is, $Fx_V \cap (x_V + V) \neq \emptyset$.

Remarks. 1. Note that if F is compact, then, by putting $\overline{F(X)} = K$, the coercivity condition (b) holds trivially.

2. Note that X does not need to have the relative topology w.r.t. E.

COROLLARY 4.1. *Let X be a convex space, E a locally convex Hausdorff t.v.s. containing X as a subset, $F \in \mathfrak{A}_c^\kappa(X, E)$, and K a nonempty compact subset of E. Suppose that*

(a) $\overline{F(X)} \cap K \subset X$; *and*

(b) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that $F(L_N) \setminus K \subset L_N$.*

Then for each $V \in \mathscr{V}$, F has a V-fixed point.

Proof. This follows from Theorem 4 with $X = D$.

Remark. Because of (a) and (b), we have $F(X) \subset X$.

COROLLARY 4.2. *Let X be a nonempty convex subset of a locally convex Hausdorff t.v.s. E. Suppose that either* (i) $F \in \mathfrak{A}_c(X, X)$ *or, more generally,* (ii) $F \in \mathfrak{A}_c^\sigma(X, X)$. *If F is compact, then F has a fixed point* $x_0 \in X$; *that is,* $x_0 \in Fx_0$.

Proof. (i) For each $V \in \mathscr{V}$, by Corollary 4.1 with $\overline{F(X)} = K$, there exist $x_V, y_V \in X$ such that $y_V \in Fx_V$ and $y_V - x_V \in V$. Since $F(X)$ is contained in the compact set K, we may assume that y_V converges to some $x_0 \in K$. Then x_V also converges to x_0. Since the graph of F is closed in $X \times K$, we have $x_0 \in Fx_0$.

(ii) Let $M = \operatorname{co} \overline{F(X)}$. Then $M \subset X$ since $\overline{F(X)} \subset X$ and X is convex. Also M is σ-compact [58, Proposition 1(3)]. Since $F \in \mathfrak{A}_c^\sigma(X, M)$, there exists an $\tilde{F} \in \mathfrak{A}_c(M, M)$ such that $\tilde{F}x \subset Fx$ for each $x \in M$. Therefore, by (i), $\tilde{F}$ has a fixed point $x_0 \in M$; that is, $x_0 \in \tilde{F}x_0 \subset Fx_0$.

This completes our proof.

Particular forms. Corollary 4.2 is due to Park [77, Theorems 3(iii) and 4] and extends many known fixed point theorems for locally convex Hausdorff t.v.s. as follows:

1. For $\mathbb{C}$ instead of $\mathfrak{A}_c$, Corollary 4.2 reduces to Hukuhara [44] which includes earlier well-known results of Brouwer [14], Schauder [86, 87], Tychonoff [103], Mazur [60], and Singbal [94].
2. For $\mathbb{K}$ instead of $\mathfrak{A}_c$, Corollary 4.2 reduces to Himmelberg [38], which extends Kakutani [50], Hukuhara [44], Bohnenblust and Karlin [12], Fan [26], and Glicksberg [30].
3. For $\mathbb{K}_c$ instead of $\mathfrak{A}_c$, Corollary 4.2 is due to Simons [92], Lassonde [56], and Ben-El-Mechaiekh [5].
4. For $\mathbb{K}_c^\sigma$ instead of $\mathfrak{A}_c^\sigma$, Corollary 4.2 is due to Lassonde [58].
5. For the class $\mathbb{A}$ of approachable maps instead of $\mathfrak{A}_c$, Corollary 4.2 is due to Ben-El-Mechaiekh and Deguire [7, 8].
6. For $\mathbb{V}_c$ instead of $\mathfrak{A}_c$, Corollary 4.2 is due to Powers [85] and Park [70].
7. For $\mathbb{V}_c^\sigma$ instead of $\mathfrak{A}_c^\sigma$, Corollary 4.2 is due to Park *et al.* [81].

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