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A GENERAL COINCIDENCE THEOREM ON CONTRACTIBLE SPACES

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ABSTRACT. We obtain a general coincidence theorem for multifunctions in very large classes defined on contractible spaces. Our theorem generalizes a recent result due to Tarafdar and Yuan (1994) and many other earlier works including the Fan-Browder fixed point theorem.

0. INTRODUCTION

Recently, Tarafdar and Yuan [TY] obtained a very interesting coincidence theorem for compact upper semicontinuous maps with contractible values defined on contractible spaces. This generalizes results on convex-valued maps defined on convex spaces.

In the present paper, we show that this result holds for non-compact maps in very general classes. Our new theorem includes a large number of known results. Our argument is based on an earlier work of the first author and H. Kim [PK2, Theorem 1], which simplifies the proof in [TY].

A *multifunction* or *map* $F : X \multimap Y$ is a function from a set X into the power set of a set Y. Note that $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$.

A triple $(X, D; \Gamma)$ is called an *H-space* [P1], [PK2] if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$; here $\langle D \rangle$ denotes the set of all nonempty finite subsets of D. The triple is called a *c-space* [H] whenever $X = D$. A convex space X in the sense of Lassonde [L] is an example of c-spaces with $\Gamma_A = \operatorname{co} A$, the convex hull of $A \in \langle X \rangle$.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of all maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of all finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and nonempty compact-valued; and

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(iii) each $F \in \mathfrak{A}_c(P,P)$ has a fixed point, where P is a convex hull of any $N \in \langle E\rangle$ for a topological vector space E.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ-values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers map $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps in topological vector spaces due to Ben-El-Mechaiekh, admissible maps of Górniewicz, and the class of permissible maps of Dzedzej. See [P2]–[P5], [PK1], [PK2].

Further, we define the following:

$F \in \mathfrak{A}_c^\sigma(X,Y) \iff$ for any σ-compact subset K of X, there is a $T \in \mathfrak{A}_c(K,Y)$ such that $Tx \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any compact subset K of X, there is a $T \in \mathfrak{A}_c(K,Y)$ as above.

Note that $\mathbb{K}_c^\sigma$ is due to Lassonde and $\mathbb{V}_c^\sigma$ to Park, Singh, and Watson. See [PSW].

Moreover, the class of approximable maps due to Ben-El-Mechaiekh and Idzik [BI] is an example of $\mathfrak{A}_c^\kappa$. The functional values of approximable maps can be convex, contractible, decomposable, or ∞-proximally connected whenever the domains of the maps are convex subsets of a locally convex Hausdorff topological vector space [BI].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any map belonging to a class in $\mathfrak{A}_c^\kappa$ is called *admissible*. For details, see [P2]–[P5], [PK1], [PK2].

The following is due to the first author and H. Kim [PK2, Theorem 1]:

Theorem 1. *Let $(X,D;\Gamma)$ be an H-space, Y a Hausdorff space, K a nonempty compact subset of Y, $F \in \mathfrak{A}_c^\kappa(X,Y)$, $S : D \multimap Y$ and $T : X \multimap Y$. Suppose that*

(1) *for each $x \in D$, Sx is compactly open in Y;*
(2) *for each $y \in F(X)$, $M \in \langle S^- y\rangle$ implies $\Gamma_M \subset T^- y$;*
(3) *$\overline{F(X)} \cap K \subset S(D)$; and*
(4) *$Y\backslash K \subset S(N)$ for some $N \in \langle D\rangle$.*

Then F and T have a coincidence point $\overline{x} \in X$; that is, $F\overline{x} \cap T\overline{x} \neq \emptyset$.

From Theorem 1, we deduce the following:

Theorem 2. *Let X be a contractible space, Y a Hausdorff space, K a nonempty compact subset of Y, and $F \in \mathfrak{A}_c^\kappa(X,Y)$. Suppose that a map $G : X \multimap Y$ satisfies the following:*

(a) *Gx is open in Y for each $x \in X$;*
(b) *for each open subset O of Y, the set $\bigcap_{y\in O} G^- y$ is empty or contractible;*
(c) *$\overline{F(X)} \cap K \subset G(X)$; and*
(d) *$Y\backslash K \subset G(N)$ for some $N \in \langle X\rangle$.*

Then F and G have a coincidence point.

Proof. Since $\overline{F(X)} \cap K$ is compact, by (a) and (c), there exists an $L \in \langle X\rangle$ such that $\overline{F(X)} \cap K \subset G(L)$. Let $D = L \cup N$. Now we define an H-space $(X,D;\Gamma)$ as follows: For any $J \in \langle D\rangle$, let

$$\Gamma_J = \begin{cases} \bigcap\{G^- y : y \in \bigcap_{x\in J} Gx\} & \text{if } \bigcap_{x\in J} Gx \neq \emptyset, \\ X & \text{otherwise,} \end{cases}$$

as in [TY]. Note that if $y \in \bigcap_{x\in J} Gx$, then $J \in \langle G^- y\rangle$. Therefore, if $O = \bigcap_{x\in J} Gx \neq \emptyset$, then $\Gamma_J = \bigcap_{y\in O} G^- y$ is nonempty and contractible by (b). It is clear that $\Gamma_J \subset \Gamma_{J'}$ whenever $J \subset J' \in \langle D\rangle$.

We now apply Theorem 1 with $S = G|_D$ and $T = G$. Then (1) and (3) follow from (a) and (c), respectively. We show that (2) holds. In fact, for each $y \in Y$ and $M \in \langle S^- y\rangle = \langle D \cap G^- y\rangle$, we have $y \in \bigcap_{x\in M} Gx \neq \emptyset$. Hence $\Gamma_M = \bigcap\{G^- z : z \in \bigcap_{x\in M} Gx\} \subset G^- y = T^- y$. Finally, (4) is same as (d). Therefore, by Theorem 1, F and G have a coincidence point. □

Remarks. 1. The open sets in (a) and (b) can be replaced by compactly open sets without affecting the conclusion of Theorem 2.

2. If X is a convex space in Theorem 2, then condition (b) can be replaced by

(b)′ for each $y \in Y$, $G^- y$ is convex.

In this case, Theorem 2 reduces to a result equivalent to Park [P4, Theorem 5], which extends many known theorems and has numerous applications in the KKM theory, as shown in [P4].

3. If F is a compact map, then by putting $Y = K = \overline{F(X)}$, condition (d) holds trivially. In this case, we have the following:

Theorem 3. *Let X be a contractible space, Y a Hausdorff compact space, and $A \in \mathfrak{A}_c^\kappa(X,Y)$. Suppose that $B : Y \multimap X$ is such that*

(a) *$B^- x$ is open for each $x \in X$ and By is nonempty for each $y \in Y$; and*

(b) *for each open set O in Y, the set $\bigcap_{y\in O} By$ is empty or contractible.*

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Ax_0$ and $x_0 \in By_0$.

Remarks. 1. When A is an u.s.c. map with nonempty compact contractible values, Theorem 3 reduces to the result of [TY].

2. As we observed in [PK2], if $F = f$ is a single-valued continuous map, the Hausdorffness assumption on Y in Theorems 1-3 is not necessary. Moreover, if $X = Y$ and $A = F = 1_X$, the identity map on X, each of Theorems 1-3 reduces to fixed point theorems. The following is a simple consequence of Theorem 3.

Theorem 4. *Let X be a compact contractible space and $G : X \multimap X$ a map satisfying*

(a) *Gx is open for each $x \in X$ and $G^- y$ is nonempty for each $y \in Y$; and*

(b) *for each open set O in X, the set $\bigcap_{y\in O} G^- y$ is empty or contractible.*

Then G has a fixed point.

Remark. If X is a convex space in Theorem 4 and if we replace (b) by (b)′, then Theorem 4 reduces to the well-known Fan-Browder fixed point theorem [F], [Br], which has numerous applications.

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