FIXED POINTS OF ACYCLIC MAPS ON TOPOLOGICAL VECTOR SPACES

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Abstract. Sufficient conditions for the existence of fixed points of acyclic maps defined on a convex subset of a topological vector space E on which E^* separates points are obtained. Main consequences are acyclic versions of fixed point theorems due to Fan, Halpern and Bergman, Himmelberg, Reich, Granas and Liu, and many others.

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In this paper, we obtain mainly sufficient conditions for the existence of fixed points of acyclic maps defined on a convex subset of a topological vector sapce E on which its topological dual E^* separates points. Such class of spaces properly contains locally convex Hausdorff topological vector spaces. Our arguments are based on a geometric property of a convex set and a variational inequality related to acyclic maps. Our main consequences are acyclic versions of well-known fixed point theorems due to Fan [**F2,3**], Halpern and Bergman [**HB**], Himmelberg [**H**], Reich [**R2**], Granas and Liu [**GL**], and many others.

A *convex space* is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

For topological spaces X and Y, a multifunction $T : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each open subset G of Y, the set

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$\{x \in X \mid Tx \subset G\}$ is open in X. Here, 2^Y denotes the class of nonempty subsets of Y. We introduce two classes of multifunctions $T : X \to 2^Y$ as follows [**GL**]:

$T \in \mathbb{K}(X, Y) \Longleftrightarrow T$ is an u.s.c. multifunction with nonempty compact convex values, where Y is a convex space.

$T \in \mathbb{V}(X, Y) \Longleftrightarrow T$ is an acyclic map; that is, an u.s.c. multifunction with compact acyclic values.

As usual, the set $\{(x, y) \mid y \in Tx\}$ is called either the graph of T, or, simply, T. Therefore $(x, y) \in T$ if and only if $y \in Tx$.

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$ on a convex set X in a vector space is *quasi-concave* [resp. *quasi-convex*] whenever $\{x \in X \mid fx > r\}$ [resp. $\{x \in X \mid fx < r\}$] is convex for each $r \in \overline{\mathbb{R}}$.

For other terminology and notations, we follow [**P4,5**].

We begin with the following geometric property with respect to an acyclic map, which is a particular form of [**P4**, Theorem 5].

Theorem 1. *Let X be a convex space, Y a Hausdorff space, $T \in \mathbb{V}(X, Y)$ a compact multifunction, $A \subset B$ sets, and $f : X \times Y \to B$ a function. Suppose that*

(1.1) *for each $x \in X$, $\{y \in Y \big| f(x, y) \in A\}$ is open; and*

(1.2) *for each $y \in T(X)$, $\{x \in X \big| f(x, y) \in A\}$ is convex.*

Then either

(a) *there exists a $\hat{y} \in \overline{T(X)}$ such that $f(x, \hat{y}) \notin A$ for all $x \in X$, or*

(b) *there exists an $(\hat{x}, \hat{y}) \in T$ such that $f(\hat{x}, \hat{y}) \in B$.*

Remarks. 1. Theorem 1 is actually a consequence of the Lefschetz fixed point theory and equivalent to Granas and Liu [**GL**, Theorem 5.1]. If $X = Y$ is compact, $A = B \subset X \times X$, $f = 1_{X \times X}$, and $T = 1_X$, then Theorem 1 reduces to the 1961 "geometric" lemma of Ky Fan [**F1**] or the Fan-Browder fixed point theorem. For the literature, see [**P4**].

2. If T is single-valued, then Y is not necessarily Hausdorff. See [**P4**].

As in [**P5**], from Theorem 1, we obtain the following variational inequality with a lopsided saddle point.

Theorem 2. ([**P5**, Theorem 2]). *Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathbb{V}(X,Y)$. Let $g : X \times Y \to \mathbb{R}$ be a continuous function such that for each $y \in Y$, $x \mapsto g(x,y)$ is quasi-convex on X. Then there exists an $(x_0,y_0) \in T$ such that*

$$g(x_0,y_0) \le g(x,y_0) \quad \textit{for all} \quad x \in X.$$

A direct consequence of Theorem 2 is as follows:

Theorem 3. *Let E be a metric topological vector space where the metric d on E has been chosen so that balls are convex, X a compact convex subset of E, and $T \in \mathbb{V}(X,E)$. Then there exists an $(x_0,y_0) \in T$ such that*

$$d(x_0,y_0) \le d(x,y_0) \quad \textit{for all} \quad x \in X.$$

Further, if $T \in \mathbb{V}(X,X)$, then T has a fixed point.

Proof. In view of Theorem 2 with $g = d$, it suffices to show that, for each $y \in E$, $x \mapsto d(x,y)$ is quasi-convex on X. In fact, for each real λ,

$$\{x \in X \mid d(x,y) < \lambda\} = X \cap \{x \in E \mid d(x,y) < \lambda\}$$

is convex. This shows the first part. The second part is trivial. □

Remark. Note that certain axiom of the metric d is not necessary. If T is single-valued, Theorem 3 reduces to Cellina [**C**], Fan [**F2**], Rassias [**Ra**], and Park [**P2**], which in turn generalize the well-known fixed point theorems of Brouwer and Schauder. Moreover, for $T \in \mathbb{K}(X,X)$, Theorem 3 generalizes other well-known theorems of Kakutani and Bohnenblust-Karlin. See [**P1-4**].

For a subset X of a topological vector space E and $x \in X$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) := \{x + r(u-x) \in E \mid u \in X,\ r > 0\},$$
$$O_X(x) := \{x - r(u-x) \in E \mid u \in X,\ r > 0\}.$$

The closures of $I_X(x)$ and $O_X(x)$ are denoted by $\overline{I}_X(x)$ and $\overline{O}_X(x)$, resp. In the sequel, $W(x)$ denotes either $\overline{I}_X(x)$ or $\overline{O}_X(x)$.

Let $\mathcal{P}$ denote the family of all continuous seminorms on a topological vector space E.

As another application of Theorem 2, we obtain the following Ky Fan type fixed point theorem for acyclic maps as in [**P5**]:

Theorem 4. *Let X be a compact convex subset of a topological vector space E on which E^* separates points and $T \in \mathbb{V}(X, E)$. Then either T has a fixed point or there exist an $(x_0, y_0) \in T$ and a $p \in \mathcal{P}$ such that*

$$0 < p(x_0 - y_0) \leq p(x - y_0) \quad \textit{for all} \quad x \in W(x_0).$$

Proof. Suppose that T has no fixed point. Then, for each $x \in X$, the origin of E does not belong to the compact set $K := x - Tx$. For each $z \in K$ there exists a linear functional $\ell_z \in E^*$ such that $\ell_z(z) \neq 0$. Since ℓ_z is continuous, there exists an open neighborhood U_z of z such that $\ell_z(y) \neq 0$ for every $y \in U_z$. Let $\{U_{z_1}, \cdots, U_{z_n}\}$ be a finite subcover of the cover $\{U_z\}_{z \in K}$ of K and

$$p_x(y) := \sum_{i=1}^{n} |\ell_{z_i}(y)| \quad \text{for each} \quad y \in E.$$

Then $p_x \in \mathcal{P}$ such that $p_x(z) > 2\delta_x$ for all $z \in K$ for some $\delta_x > 0$.

Since T is u.s.c., there exists an open neighborhood V_x of $x \in X$ such that $p_x(u - v) > \delta_x$ for all $u \in V_x$ and $v \in Tu$. Since $\{V_x \,|\, x \in X\}$ covers X and X is compact, there exists a finite subcover $\{V_{x_1}, \ldots, V_{x_k}\}$ of X. Let $p := \max\{p_{x_i} \,|\, 1 \leq i \leq k\}$ and $\delta = \min\{\delta_{x_i} \,|\, 1 \leq i \leq k\} > 0$. Then $p \in \mathcal{P}$ and $p(x - y) > \delta$ for all $(x, y) \in T$.

We define a function $g : X \times Y \to \mathbb{R}$ by $g(x, y) = p(x - y)$ for $(x, y) \in X \times Y$, where $Y := T(X)$ is compact. Then clearly g and T satisfy all of the requirements of Theorem 2. Therefore there exists an $(x_0, y_0) \in T$ such that

$$0 < p(x_0 - y_0) \leq p(x - y_0) \quad \text{for all} \quad x \in X.$$

For the remainder of the proof, just follow that of [**P5**, Theorem 3].

Remarks. 1. For a locally convex Hausdorff topological vector space E, Theorem 4 reduces to [**P5**, Theorem 3], which extends earlier works of Fan, Reich, Ha, and Park.

2. Note that the x_0 in the conclusion of Theorem 4 belongs to the boundary of X, Bd X. In fact, suppose that $x_0 \in \operatorname{Int} X$. Then x_0 is an internal point and $W(x_0) = E$. By putting $x = y_0$, we have $0 < p(x_0 - y_0) \leq 0$ in the conclusion of Theorem 4, which is a contradiction.

As a direct consequence of Theorem 4, we have the following as in [**P5**]:

Theorem 5. *Let X be a compact convex subset of a topological vector space E on which E^* separates points and $T \in \mathbb{V}(X, E)$. If T satisfies one of the following conditions, then T has a fixed point.*

For each $x \in Bd\, X$,

(0) *for each $y \in Tx$ and each $p \in \mathcal{P}$, $p(y - x) > 0$ implies $p(y - x) > p(y - z)$ for some $z \in W(x)$.*

(i) *for each $y \in Tx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that*

$$|\lambda| < 1 \text{ and } \lambda x + (1 - \lambda)y \in W(x).$$

(ii) $Tx \subset W(x)$.

(iii) *for each $y \in Tx$, there exists a number λ (as in (i)) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda x + (1 - \lambda)y \in X.$$

(iv) $Tx \subset IF_X(x) := \{x + c(u - x) \,|\, u \in X,\ Re(c) > 1/2\}$.

(iv)′ $Tx \subset OF_X(x) := \{x + c(u - x) \,|\, u \in X,\ Re(c) < -1/2\}$.

(v) $Tx \subset X$.

(vi) $T(X) \subset X$.

Remarks 1. For locally convex spaces, Theorem 5 reduces to [**P5**, Theorem 4], which extends well-known earlier theorems of Brouwer, Schauder, Tychonoff, Rothe, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Halpern, Browder, Reich, Fitzpatrick and Petryshyn, Ha, and others. For details, see [**P5**]. For a topological vector space E on which E^* separates points, Theorem 5 for $\mathbb{K}$ instead of $\mathbb{V}$ includes Fan [**F3**, Corollaire 2], Halpern and Bergman [**HB**, Theorems 4.1 and 4.3], Kaczynski [**Ka**, Théorèmes 1-4], Granas and Liu [**GL**, Theorem 10.5], and Park [**P1**, Theorems 6 and 8].

2. If $T \in \mathbb{K}(X,E)$, then more general conditions than (0)-(vi) suffice for the existence of fixed points. See [**P3,4**]. In this case, e.g., (ii) can be replaced by $Tx \cap W(x) \neq \emptyset$. However, this is not true for $T \in \mathbb{V}(X,E)$.

Example. Let $E = \mathbb{R}^2$, $X = [-1,1] \times \{0\}$, and $T \in \mathbb{V}(X,E)$ such that, for each $x \in X$, Tx is the union of two segments joining $(-2,0)$ and (0,1), (0,1) and (2,0). Then T is a constant acyclic map and $Tx \cap W(x) \neq \emptyset$ for $x \in X$. However, T has no fixed point.

For the outward case (ii), we have the following surjectivity result:

Theorem 6. *Let X, E, and T be the same as in Theorem 5. If $Tx \subset \overline{O}_X(x)$ for each $x \in Bd\,X$, then T has a fixed point and $T(X) \supset X$.*

Proof. By Theorem 5(ii), T has a fixed point. Suppose $X \not\subset T(X)$. We may assume that the origin 0 is a point of $X \backslash T(X)$. The complement U of $T(X)$ is a neighborhood of 0, so we can choose $c > 1$ such that $cU \supset X$. Then $cT(X)$ is disjoint from X, and so the map cT can have no fixed point. However, since the weakly outward set $\overline{O}_X(x)$ is closed under the multiplication by a constant $c > 1$ ([**HB**, Lemma 4.2]), $cTx \subset \overline{O}_X(x)$ for all $x \in X$. This is a contradiction. □

Remark. We followed the proof of [**HB**, Theorem 4.3], which is the single-valued case of Theorem 6.

As an application of Theorem 5, we have the following:

Theorem 7. *Let E be a topological vector space on which E^* separates points and K a nonempty compact convex subset of E. Then any continuous affine map $f : K \to E$ such that $K \subset fK$ has a fixed point.*

Proof. Let $T = f^{-1} : fK \to 2^K$. Then $T \in \mathbb{K}(fK, fK)$. Therefore, by

Theorem 5(vi), there exists an $x \in K$ such that $x \in f^{-1}x$; that is, $x = fx$. □

From Theorem 1, we obtain the following acyclic version of the Himmelberg fixed point theorem.

Theorem 8. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E. If $T \in \mathbb{V}(X, X)$ is compact, then T has a fixed point.*

Proof. Let Y be a compact set such that $T(X) \subset Y \subset X$, and V an open convex neighborhood of the origin in E. We apply Theorem 1 with $A = V$, $B = E$, and $f : X \times Y \to E$ defined by $f(x, y) = y - x$ for $(x, y) \in X \times Y$. Then all of the requirements are satisfied. In fact,

(1.1) for each $x \in X$, $\{y \in Y \mid f(x, y) \in A\} = (x + V) \cap Y$ is open in Y; and

(1.2) for each $y \in Y$, $\{x \in X \mid f(x, y) \in A\} = (y - V) \cap X$ is convex.

Moreover, Property (a) of Theorem 1 does not hold. For, since $Y \subset X$, for every $y \in Y$, there exists an $x \in X$ such that $y \in x + V$; that is, $f(x, y) = y - x \in V$. Therefore Property (b) holds; that is, there exists an $x_V \in X$ such that $Tx_V \cap (x_V + V) \neq \emptyset$. Since $T(X)$ is relatively compact and the graph of T is closed in $X \times Y$, the following well-known lemma implies the conclusion. □

Lemma. [**L**] *Let X be a nonempty convex subset of a Hausdorff topological vector space E and $T : X \to 2^X$ a compact u.s.c. multifunction with nonempty closed values. Then T has a fixed point if and only if, for every neighborhood V of the origin of E, there exists a point $x_V \in X$ such that $Tx_V \cap (x_V + V) \neq \emptyset$.*

Remark. If $T \in \mathbb{K}(X, X)$, then Theorem 8 reduces to Himmelberg [**H**, Theorem 2], which extends earlier works of Schauder, Mazur, Bohnenblust and Karlin, Hukuhara, Singbal, Tychonoff, Kakutani, Fan, and Glicksberg. An extended version of Theorem 8 was given in [**P4**, Theorem 7]. Recently, Kum [**Ku**] obtained applications of Theorem 8 to generalized quasi-variational inequalities.

As an application of Theorem 8, we obtain the following acyclic version of Reich's theorem on condensing maps with the Leray-Schauder boundary condition. For the definition of condensing multifunctions, see Su and Sehgal [**SS**].

Theorem 9. *Let C be a nonempty closed subset of a locally convex Hausdorff topological vector space E and $T : C \to 2^E$ an u.s.c. multifunction with nonempty closed acyclic values. Suppose that T has a bounded range, and that there is a point $w \in \mathrm{Int}\, C$ such that*

(L-S) *for every $x \in \mathrm{Bd}\, C$ and $y \in Tx$,*

$$y - w \neq m(x - w) \quad \text{for all} \quad m > 1.$$

Then F has a fixed point if one of the following holds:

(i) *T is compact.*
(ii) *T is condensing and E is quasi-complete.*
(iii) *T is condensing with compact values and C is quasi complete.*

Proof. Just follow the proof of Reich [**R2**] and use Theorem 8 instead of Himmelberg [**H**, Theorem 2]. □

Remark. If T has convex values, then Theorem 9 reduces to Reich [**R2**, Theorem]. Reich [**R1**] noted that his earlier version of Theorem 9 holds for a metrizable E. However, we do not assume the metrizability of E in Theorem 9. Recently, Ben-El-Mechaiekh [**B**] gave a proof of Theorem 9(i) for $T \in \mathbb{K}(C, E)$ based on a matching theorem formulated by Ky Fan [**F4**].

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