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FIVE EPISODES RELATED TO THE FAN-BROWDER FIXED POINT THEOREM

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Abstract. From a Fan-Browder type coincidence theorem due to the author [P4], we deduce improved versions of results of Sehgal, Singh and Watson [SSW] and Ben-El-Mechaiekh and Kryszewski [BK]. We discuss mutual relations between existence theorems of maximizable u.s.c. quasiconcave functions on convex spaces and their consequences. Moreover, a Ha type coincidence theorem for the Fan-Browder maps is proved. Finally, we give a number of partial solutions of a fixed point problem raised by Ben-El-Mechaiekh [B2,4] and several open problems.

0. Introduction

In [F], using his own generalization of the classical Knaster-Kuratowski-Mazurlinebreak kiewicz theorem (simply, KKM theorem), Ky Fan established an elementary but very basic "geometrical" lemma for multimaps. Later, Browder [Br] restated this result in the more convenient form of a fixed point theorem by means of the Brouwer fixed point theorem and the partition of unity argument. Since then this result is known to be the Fan-Browder fixed point theorem, and has been applied to various fields in mathematical sciences. For the literature, see [P4].

Recently, in [P4], a coincidence theorem extending the Fan-Browder theorem for composites of multimaps in very general classes was obtained, and applied to deduce various fundamental theorems in the KKM theory. Actually, the results in [P4] extend, improve, and unify main theorems in more than one hundred published works.

Our aim in the present paper is to show that our theorem can be used to obtain improved versions of other works. As we suggested in the title, five sections from 2 to 6 cover mutually rather independent topics.

In Section 2, we show that the main coincidence theorem of Sehgal, Singh, and Watson [SSW1,2] is actually equivalent to the Fan-Browder theorem.

Section 3 deals with the results of Ben-El-Mechaiekh and Kryszewski [BK], which are shown to be consequences of those in [P4].

In Section 4, we discuss mutual relations between existence theorems of maximizable u.s.c. quasiconcave functions on convex spaces due to Park and Bae [PB1] and to [BK]. Moreover, related results appeared in [PB2], [KL], and [D] are also discussed.

Section 5 deals with a Ha type coincidence theorem [H], which is an affirmative solution of a question of Ben-El-Mechaiekh [B3].

In Section 6, we consider the problem whether the Fan-Browder theorem holds for compact maps instead of assuming compactness of the domain. This was also raised by Ben-El-Mechaiekh [B2,4]. We give several partial solutions to this problem and some open problems.

1. Preliminaries

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of a set Y; that is, a function with the *values* $Tx \subset Y$ for $x \in X$ and the *fibers* $T^- y = \{x \in X : y \in Tx\}$ for $y \in Y$. For $A \subset X$, let $T(A) = \bigcup\{Tx : x \in A\}$, and for $B \subset Y$, $T^-(B) = \{x \in X : Tx \cap B \neq \emptyset\}$.

Let X be a set (in a vector space), D a nonempty subset of X, and $\langle D \rangle$ the set of all nonempty finite subsets of D. Then (X, D) is called a *convex space* if the convex hull $\operatorname{co} N$ of any $N \in \langle D \rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. Such convex hulls are called *polytopes*. A subset A of X is said to be *D-convex* if, for any $N \in \langle D \rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [L1].

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of all maps $F : X \to 2^Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of all finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and nonempty compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in topological vector spaces, admissible maps in the sense of Górniewicz, permissible maps of Dzedzej, and others. Moreover, we define

$F \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

A class $\mathfrak{A}_c^\kappa$ is said to be *admissible*. Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [L2] and $\mathbb{V}_c^\sigma$ due to Park, Singh, and Watson. Note that $\mathbb{K}_c^\sigma$ includes classes $\mathbb{K}$, $\mathbb{R}$, and $\mathbb{T}$ in [L2]. The approximable maps recently due to Ben-El-Mechaiekh and Idzik [BI] belong to $\mathfrak{A}_c^\kappa$. Therefore, any compact-valued u.s.c. map $F : X \to 2^E$, where E is a locally convex Hausdorff topological vector space and $X \subset E$, belongs to $\mathfrak{A}_c^\kappa$ if its values are all convex, contractible, decomposable, or ∞-proximally connected. See [BI].

For the literature on admissible classes, see [PK]. For other terminology, we follow mainly [P3,4].

The following is basic in this paper:

THEOREM 0. [P4, Theorem 5] *Let (X, D) be a convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $x \in D$, $Sx \subset Tx$ and Sx is compactly open;*
(2) *for each $y \in F(X)$, $T^- y$ is D-convex;*
(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*
(4) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that $F(L_N) \backslash K \subset S(L_N \cap D)$.*

Then F and T have a coincidence point $x_0 \in X$; that is, $Fx_0 \cap Tx_0 \neq \emptyset$.

It is known that if F is single-valued, the Hausdorffness of Y is not necessary. Note that if N is empty, then condition (4) holds trivially with $L_N = \emptyset$.

If $X = Y = D = K$ and $F = 1_X$, then Theorem 0 reduces to the following form of the Fan-Browder fixed point theorem:

THEOREM 1. [P4, Theorem 2] *Let X be a compact convex space and $G : X \multimap X$ a map satisfying*

(1) *for each $y \in X$, $G^- y$ is convex; and*
(2) *$\{\operatorname{Int} Gx : x \in X\}$ covers X.*

Then G has a fixed point $x_0 \in X$; that is, $x_0 \in Gx_0$.

2. A Sehgal-Singh-Watson type theorem

In [SSW1,2], the authors obtained a coincidence theorem and its direct consequences. In this section, we show that improved versions of their results can be deduced from Theorem 1.

THEOREM 2. *Let X be a convex space, Y a topological space, and $T, S : X \multimap Y$ maps satisfying the following:*

(1) *T is a compact l.s.c. map with nonempty values; and*
(2) *S is an open-valued map such that $S^- y$ is nonempty for each $y \in \overline{T(X)}$ and $(S^- T)x$ is convex for each $x \in X$.*

Then T and S have a coincidence point.

Proof. For each $y \in \overline{T(X)}$, there exists an $x \in S^- y$ by (2). Since $y \in Sx$, we have $\overline{T(X)} \subset \bigcup_{x \in X} Sx$. Since each Sx is open and $\overline{T(X)}$ is compact, we have $T(X) \subset \bigcup_{i=1}^n Sx_i$ for some $\{x_1, \ldots, x_n\} \in \langle X \rangle$. Therefore, we have $X \subset \bigcup_{i=1}^n (T^- S)x_i$.

Let $M = \operatorname{co}\{x_1, \ldots, x_n\}$ and define a map $G : M \multimap M$ by $Gx = (T^- S)x \cap M$ for $x \in M$. We show that G satisfies the requirements of Theorem 1 with $X = M$.

(i) For each $z \in M$,

$$\begin{aligned} G^- z &= \{x \in M : z \in Gx\} = \{x \in M : z \in (T^- S)x \cap M\} \\ &= \{x \in M : Tz \cap Sx \neq \emptyset\} = \{x \in M : x \in (S^- T)z\} \\ &= (S^- T)z \cap M. \end{aligned}$$

Since $Tz \neq \emptyset$, $(S^- T)z$ is nonempty and convex by (2). Therefore, $G^- z$ is convex in M, and condition (1) of Theorem 1 is satisfied.

(ii) Since $X \subset \bigcup_{i=1}^n (T^- S)x_i$, we have

$$M = X \cap M \subset \bigcup_{i=1}^n [(T^- S)x_i \cap M] = \bigcup_{i=1}^n Gx_i.$$

We show that each Gx_i is open in M. In fact, for each i, $Gx_i = (T^- S)x_i \cap M$. Since Sx_i is open and T is l.s.c., $(T^- S)x_i$ is open in X and hence Gx_i is open in M. Therefore, condition (2) of Theorem 1 holds.

Since M is a compact convex space, by Theorem 1, G has a fixed point $x_0 \in M$; that is, $x_0 \in (T^- S)x_0$ or $Tx_0 \cap Sx_0 \neq \emptyset$. This completes our proof.

Note that Theorem 2 improves Sehgal *et al.* [SSW1, 2; Theorem 2], where X is assumed to be a *closed* convex subset of a Hausdorff topological vector space and Y is also a Hausdorff topological vector space. In view of Theorem 2, [SSW1,2; Corollaries 1 and 2] can be also improved.

From Theorem 2 we have the following Fan-Browder theorem [Br]:

THEOREM 3. *Let X be a compact convex space and $S : X \multimap X$ an open-valued map with nonempty convex fibers. Then S has a fixed point.*

Proof. Put $X = Y$ and $T = 1_X$ in Theorem 2.

It is well known that Theorem 3 is equivalent to Theorem 1, from which we deduced Theorem 2. Therefore, Theorems 1-3 are mutually equivalent.

3. A Ben-El-Mechaiekh and Kryszewski type theorem

In [BK], the authors gave a generalization of the Fan-Browder theorem and applied it to equilibrium for perturbations of multimaps. We show that their result follows from our Theorem 0.

THEOREM 4. [BK, Theorem 4.1] *Let X be a convex space and $\Phi : X \multimap X$ a map satisfying the following:*

(1) $\forall y \in X,\ \Phi^- y$ *is open;*
(2) $\forall x \in X,\ \Phi x$ *is nonempty convex;*
(3) *there exists a compact subset K of X such that for any finite subset N of X there exists a compact convex subset C_N of X containing N such that $\Phi x \cap C_N \neq \emptyset$, $\forall x \in C_N \backslash K$.*

Then $\exists x_0 \in X$ such that $x_0 \in \Phi x_0$.

Proof. We apply Theorems 0 and 1.

Case 1. If $K = \emptyset$, choose any $N \in \langle X \rangle$ and define $G : C_N \multimap C_N$ by $G^- y = \Phi y \cap C_N$ for $y \in C_N$. Then $G^- y$ is nonempty convex by (2) and (3). Moreover, for each $y \in C_N$, there exists an $x \in G^- y = \Phi y \cap C_N$; that is, $y \in \Phi^- x \cap C_N = Gx$. Since Gx is open by (1), C_N is covered by $\{\operatorname{Int} Gx : x \in C_N\}$. Hence, Theorem 1 with C_N instead of X works.

Case 2. If $K \neq \emptyset$, we can use Theorem 0 with $X = D = Y$, $F = 1_X$, $S = T = \Phi^-$, and $L_N = C_N$. Then conditions (1) and (2) of Theorem 0 hold immediately. For any $x \in K$, there exists a $y \in \Phi x$ by (2), and hence $K \subset \Phi^-(X)$. Therefore, condition (3) of Theorem 0 holds. Furthermore, for condition (4) of Theorem 0, let $x \in C_N \backslash K$. Then there exists a $y \in \Phi x \cap C_N$ by (3); that is, $y \in C_N$ such that $x \in \Phi^- y$, which implies that $C_N \backslash K \subset \Phi^-(C_N)$. Hence Theorem 0 works.

Similarly, [BK, Proposition 4.2] is a simple particular form of [P4, Theorem 7]. Note that, in Theorem 4, N can be nonempty; otherwise condition (3) holds trivially with $C_N = \emptyset$.

4. Existence of maximizable quasiconcave functions

In [B], Bellenger gave a theorem on the existence of maximizable quasiconcave real functions defined on paracompact convex spaces. His theorem is a common generalizations of earlier works of Fan and Simons. Bellenger also raised as an open question whether the paracompactness condition is necessary in his theorem. Park and Bae [PB1] gave the affirmative answer to this question and applied it to obtain the Fan type nonseparation theorem and some coincidence or fixed point theorems. Those were refined and generalized in later works of Park *et al.* [P2, PB2, PBK].

Recently, a related result is due to Ben-El-Mechaiekh and Kryszewski as follows:

THEOREM 5. [BK, Theorem 4.2] *Let X be a convex subset in a topological vector space, Y a subset in $\{\varphi : X \to \mathbf{R} \,|\, \varphi$ is u.s.c. and quasiconcave$\}$, and $\Psi : X \to 2^Y$ a multifunction. Assume that the following properties are satisfied:*

(i) *Ψ admits a continuous selection s;*
(ii) *there exists a compact subset K of X such that for each finite subset N of X, there exists a compact convex subset C_N of X containing N such that*

$$\forall x \in C_N \backslash K,\ \forall \varphi \in \Psi x,\ \varphi(x) < \max_{u \in C_N} \varphi(u).$$

Then

$$\exists x_0 \in K,\ \exists \varphi_0 \in \Psi x \text{ such that } \varphi_0(x_0) = \max_{u \in X} \varphi_0(u).$$

In [BK], the authors deduced Theorem 5 from [BK, Proposition 4.2], which is a consequence of Theorem 4. Note that Theorem 5 can be improved as follows:

(1) X can be a convex space.

(2) The set $\{\varphi : X \to \mathbf{R} : \varphi$ is u.s.c. and quasiconcave$\}$ should have a convex space topology; otherwise we can not have a "continuous" selection s.

(3) Condition (ii) can be stated for each nonempty finite subset N as for Theorem 4.

Note that, in [BK], the authors claimed incorrectly that Theorem 5 contains results of [P1]. It seems to be clear that Theorem 5 was motivated by the main result of Park and Bae [PB, Theorem 1], which was later improved by Park, Bae, and Kang [PBK] as follows:

THEOREM 6. [PBK, Theorem 3.1] *Let X be a convex space, Y a convex space consisting of u.s.c. quasiconcave real functions on X, and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, Tx is a convex subset of Y and Sx is a nonempty subset of Tx;*
(2) *for each $f \in Y$, $S^- f$ is compactly open in X; and*
(3) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash K$ and each $f \in Tx$, we have $f(x) < \max f(L_N)$.*

Then there exist an $x \in K$ and an $f \in Tx$ such that $f(x) = \max f(X)$.

If X is compact or paracompact, then Theorem 6 follows from the corrected version of Theorem 5 by using well-known selection theorems. See the next section of this paper and [B].

Now we have the following:

Problem 1. Under the situation of Theorem 6 with $S = T$, does S allow a continuous selection?

Note that the Fan-Browder fixed point theorem was used to prove [PB1, Theorem 1]. However, Theorem 6 was deduced from a very general section property of convex spaces in [PBK]. If we could solve Problem 1 affirmatively, then we would have another proof of Theorem 6 by adopting the corrected version of Theorem 5.

Just after [PB1] was published, Park and Bae [PB2] applied [PB1, Theorem 1] to obtain extended versions of the existence theorem of Ben-El-Mechaiekh [B1] on zeros for multimaps with non-compact domains, and to other problems. Motivated by [PB1], the works of Kim and Lee [KL] and Ding [D] appeared. Since these papers are closely related to [PB2], it would be interesting to compare results in these three papers:

(1) [KL, Theorem 1] is incorrectly stated.
(2) [KL, Theorem 2] is a particular form of [PB2, Corollary 1].
(3) [KL, Theorem 3] is a particular form of [PB2, Corollary 3].
(4) [D, Theorem A] is an H-space version of [PB1, Theorem 1].
(5) [D, Theorem 2.1] is an H-space version of [P3, Theorem 3; PB1, Theorem 1] and [D, Theorem 2.2] is its convex space version.

(6) [D, Theorem 2.3] is almost same to [PB2, Corollary 1].
(7) [D, Theorem 2.4] is almost same to [PB2, Corollary 2].
(8) [D, Theorem 2.5] is already given in [P3, Theorem 6].
(9) [D, Theorem 2.6] is almost same to [PB2, Corollary 3].

In (6), (7), and (8), the results in [D] are stated for the type of multimaps studied by [P3], while those in [PB2] for upper hemicontinuous multimaps.

Note that H. Kim [K, Theorem 2] independently obtained an extension of [D, Theorem A] and applied it to generalize the Tychonoff-Fan type theorem and to systems of convex inequalities. Actually, after she finished up [K], the present author informed her about [D].

5. A Ha type coincidence theorem

Browder's proof [Br] of the Fan-Browder fixed point theorem exploited the partition of unity argument in order to obtain a continuous selection. Refining his argument, the following was obtained:

LEMMA. [PBK, Theorem 2.1] *Let X be a Hausdoff space, Y a convex space, and $S, T : X \multimap Y$ multimap satisfying*

(1) *for each $x \in X$, $\emptyset \neq Sx \subset Tx$ and Tx is convex; and*
(2) *for each $y \in Y$, $S^{-}y$ is compactly open in X.*

Then, for any nonempty compact subset K of X, there exists a continuous function $f : K \to Y$ such that

(3) *$f(x) \in Tx$ for each $x \in K$;*
(4) *$f(K)$ is contained in a polytope of Y; and*
(5) *for any compact subset L of X containing K, there exists a continuous extension $\tilde{f} : L \to Y$ such that $\tilde{f}(x) \in Tx$ for each $x \in L$ and $\tilde{f}(L)$ is contained in a polytope of Y.*

From Lemma, we have the following:

THEOREM 7. *Let X and Y be convex spaces and $A, B : X \multimap Y$ multimaps such that*

(1) *A has nonempty convex values and compactly open fibers; and*
(2) *B has compactly open values and nonempty convex fibers.*

If $A(X)$ is contained in a Hausdorff compact subset K of Y, then A and B have a coincidence point $x_0 \in X$; that is, $Ax_0 \cap Bx_0 \neq \emptyset$.

Proof. Since $B^- : Y \multimap X$ has nonempty convex values and compactly open fibers and K is Hausdorff and compact, $B^-|_K$ has a continuous selection $g : K \to P$ for some polytope P of X. Since P is Hausdorff and compact, $A|_P : P \multimap K$ has a continuous selection $f : P \to K$ by Lemma. Now the continuous map $gf : P \to P$ has a fixed point $x_0 \in P \subset X$ by the Brouwer fixed point theorem; that is, $x_0 = (gf)x_0$. Therefore, $f(x_0) \in g^-(x_0) \subset Bx_0$ and $f(x_0) \in Ax_0$. This shows $Ax_0 \cap Bx_0 \neq \emptyset$.

Note that Theorem 7 is an affirmative answer to a question raised by Ben-El-Mechaiekh [B3]. Therefore, Theorem 7 extends [B3, Theorem 3].

This kind of coincidence theorems were first considered under different assumptions and applied to non-compact versions of Sion's minimax theorem by Ha [H].

6. A problem of Ben-El-Mechaiekh

It is well-known that Schauder first considered certain compactness of a continuous function instead of assuming compactness of its domain. Therefore, it seems to be natural to ask whether the Fan-Browder theorem holds for compact maps.

In this section, we begin with the following:

THEOREM 8. *Let X be a Hausdorff convex space and $A : X \multimap X$ a multimap with nonempty convex values and compactly open fibers. If A is compact, then A^n has a fixed point for $n \geq 2$.*

Proof. By Lemma, we have $A \in \mathbb{C}_c^{\kappa}(X, X)$ and hence $A^m \in \mathbb{C}_c^{\kappa}(X, X)$ for $m \geq 1$. Note that A^m is compact. Now we apply Theorem 0 with $X = Y = D$, $F = A^m$, $S = T = A^-$, and $K = \overline{A^m(X)}$. Then

(1) for each $x \in X$, $A^- x$ is compactly open;

(2) for each $y \in X$, Ay is nonempty convex;

(3) for each $y \in \overline{A^m(X)} = K$, there exists $x \in Ay$ or $y \in A^- x$ by (2), whence we have $K \subset A^-(X)$; and

(4) $A^m(X) \backslash K = \emptyset$.

Therefore, all of the requirements of Theorem 0 are satisfied. Hence, by Theorem 0, we have a coincidence point $x_0 \in X$ of A^m and A^-; that is, there exists a $y_0 \in A^m x_0 \cap A^- x_0$. Since $x_0 \in Ay_0$ and $y_0 \in A^m x_0$, we have $x_0 \in A^{m+1} x_0$. This completes our proof.

A slightly partial form of Theorem 8 was noted by Ben-El-Mechaiekh [B2,4]. Moreover, he raised the following:

Problem 2. Does A (with open fibers) have a fixed point under the hypothesis of Theorem 8?

This is still open. We discuss partial solutions of this problem.

First of all, if X is a convex subset of a locally convex Hausdorff topological vector space, then Problem 2 is affirmative. See Ben-El-Mechaiekh *et al.* [BDG, Theorem 3.2; B4, Theorem 3] and Park [P5, Corollary 2.2]. In those works, it is used that $A \in \mathbb{C}_c^\sigma$. See Lassonde [L2]. However, we had a more general result as follows:

THEOREM 9. [P4, Theorem 4] *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $A \in \mathfrak{A}_c^\sigma(X, X)$. If A is compact, then A has a fixed point.*

Here, a natural question arises as follows:

Problem 3. Can $\mathfrak{A}_c^\sigma$ be replaced by $\mathfrak{A}_c^\kappa$ in Theorem 9?

For non-locally convex case, we have two more partial solutions of Problem 2 as follows:

THEOREM 10. [P6, Theorem 2′] *Let X be a nonempty convex subset of a Hausdorff topological vector space E, and $T : X \multimap X$ a compact map such that*

(1) *for each $x \in X$, Tx is convex; and*

(2) *$\{\operatorname{Int} T^- y\}_{y \in X}$ covers X.*

If $\overline{T(X)}$ is convexly totally bounded (in the sense of Idzik [I]*) in E, then T has a fixed point.*

THEOREM 11. [P7, Theorem 2′] *Let X be a nonempty convex subset of a Hausdorff topological vector space E, and $S, T : X \multimap X$ compact maps such that*

(1) *for each $x \in X$, $\operatorname{co} Sx \subset Tx$; and*

(2) $\{\operatorname{Int} S^{-}y\}_{y\in X}$ *covers X.*

If X is admissible (in the sense of Klee [Kl]*) in E, then T has a fixed point.*

Note that there are many open questions on the concepts of convexly totally boundedness and admissibility related to the Schauder conjecture [I].

In view of Theorems 8–11, we can raise the following general form of the Schauder conjecture:

Problem 4. Does a nonempty convex subset of a topological vector space have the fixed point property for compact continuous functions? Or for $\mathfrak{A}_c^\sigma$ or $\mathfrak{A}_c^\kappa$?

For non-compact maps, the following result on convex-valued maps would be interesting:

THEOREM 12. [P2, Theorem 5] *Let X be a convex space and $S, T : X \multimap X$ maps satisfying*

(1) *for each $x \in X$, $\operatorname{co} Sx \subset Tx$;*

(2) *for each $y \in X$, $S^{-}y$ is closed; and*

(3) $X = \bigcup_{y\in A} S^{-}y$ *for some finite subset A of X.*

Then T has a fixed point.

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