# REMARKS ON ADMISSIBLE MULTIFUNCTIONS AND THE LERAY-SCHAUDER PRINCIPLES

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Abstract. We veriew some of our previous works on admissible classes of multifunctions, and deduce some new fixed point theorems and the Leray-Schauder type principles for condensing admissible multifunctions.

## 0. Introduction

Recently, in a sequence of papers [P1-18], the author introduced the admissible classes of multifunctions, which are broad enough to include most of important multifunctions appearing in nonlinear analysis and algebraic topology. For the admissible classes, we established the foundations of the KKM theory via coincidences of such multifunctions [P2,3] and the fixed point theory in topological vector spaces [P1,9].

Some of those new results were, consecutively, applied to the following topics:

(1) Best approximation problems [P4,6,11,14].
(2) Generalized variational or quasi-variational inequalities and generalized complementarity problems [P17,18, PC1-4].
(3) Generalized Leray-Schauder or Birkhoff-Kellogg theorems [P5,7,12,13,16].
(4) Extensions to generalized convex spaces [PK1-3, PJ].
(5) Applications of a generalized minimax inequality [P18].
(6) Openess of multifunctions [P15].

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In the present paper, we review some of our previous results and obtain some new results on admissible classes. In fact, we deduce some new fixed point theorems and the Leray-Schauder type principles for condensing admissible multifunctions.

Section 2 deals with the notion and examples of admissible classes of multifunctions. In Section 3, we review a coincidence theorem on generalized convex spaces and obtain some fixed point theorems on multifunctions defined on various types of spaces. Section 4 deals with fixed point results on condensing maps and a Schöneberg type theorem on admissible maps with the Leray-Schauder boundary conditions. Finally, in Section 5, we prove correct and generalized forms of the Leray-Schauder theorems due to Milojević [M].

## 1. Admissible classes of multifunctions

A *multifunction* (or *map*) $F : X \multimap Y$ is a function from a set $X$ into the power set $2^Y$ of $Y$; that is, a function with nonempty *values* $Fx \subset Y$ for $x \in X$ and *fibers* $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. A map $F : X \multimap Y$ is *compact* provided $F(X)$ is contained in a compact subset of a topological space $Y$. For any $B \subset Y$, the (*lower*) *inverse* of $B$ under $F$ is defined by

$$F^{-}(B) = \{x \in X : Fx \cap B \neq \emptyset\}.$$

Given two maps $F : X \multimap Y$ and $G : Y \multimap Z$, the *composite* $GF : X \multimap Z$ is defined by $(GF)x = G(Fx)$ for $x \in X$.

For topological spaces $X$ and $Y$, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-}(B)$ is closed in $X$.

Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact.

A *convex space* is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of all maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of all finite composites of maps in $\mathbb{X}$.

For topological spaces $X$ and $Y$, we define

$f \in \mathbb{C}(X, Y) \Longleftrightarrow f$ is a (single-valued) continuous function.

$T \in \mathbb{K}(X, Y) \Longleftrightarrow T$ is a *Kakutani map*; that is, $Y$ is a convex space and $T$ is u.s.c. with compact convex values.

$T \in \mathbb{V}(X, Y) \Longleftrightarrow T$ is an *acyclic map*; that is, $T$ is u.s.c. with compact acyclic values.

A class $\mathfrak{A}$ of maps is one satisfying

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope $P$, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are $\mathbb{C}, \mathbb{K}, \mathbb{V}$, the Aronszajn maps $\mathbb{M}$ (with $R_\delta$ values) [Gr], the O'Neill maps $\mathbb{N}$ (with values consisting of one or $m$ acyclic components, where $m$ is fixed) [Gr], admissible maps of Górniewicz [G], the class of permissible maps of Dzedzej [D], and others. Note that $\mathbb{K} \subset \mathbb{M} \subset \mathbb{V} \subset \mathbb{N}$ and those are all included in the Górniewicz classes and hence in the Dzedzej classes.

Further, we define the following:

$T \in \mathfrak{A}_c^\sigma(X, Y) \Longleftrightarrow$ for any $\sigma$-compact subset $K$ of $X$, there is a $\tilde{T} \in \mathfrak{A}_c(K, Y)$ such that $\tilde{T}x \subset Tx$ for each $x \in K$.

$T \in \mathfrak{A}_c^\kappa(X, Y) \Longleftrightarrow$ for any compact subset $K$ of $X$, there is a $\tilde{T} \in \mathfrak{A}_c(K, Y)$ as above.

The class $\mathbb{K}_c^\sigma$ due to Lassonde [L2] and $\mathbb{V}_c^\sigma$ due to Park, Singh, and Watson [PSW] belong to $\mathfrak{A}_c^\sigma$. Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. For details, see [PK1]. Any class in $\mathfrak{A}_c^\kappa$ is said to be *admissible*.

In this paper, t.v.s. means Hausdorff topological vector spaces. Let $X$ and $Y$ be subsets of t.v.s. $E$ and $F$, respectively. Given two open neighborhoods $U$ and $V$

of the origins in $E$ and $F$, respectively, a $(U,V)$-*approximate continuous selection* of a map $T : X \multimap Y$ is an $s \in \mathbb{C}(X,Y)$ satisfying

$$s(x) \in (T[(x+U)\cap X] + V) \cap Y \quad \text{for} \quad x \in X.$$

$T \in \mathbb{A}(X,Y) \iff T$ is u.s.c., compact-valued, and *approachable*; that is, $T$ admits a $(U,V)$-approximative continuous selection for every neighborhoods $U$ and $V$ of the origins in $E$ and $F$, respectively. See [BD1-3].

$T \in \mathbb{A}^\kappa(X,Y) \iff T$ is *approximable*; that is, for any compact subset $K$ of $X$, there is a $\tilde{T} \in \mathbb{A}(K,Y)$ such that $\tilde{T}x \subset Tx$ for each $x \in K$. See [BI].

Note that $\mathbb{A} = \mathbb{A}_c$ and $\mathbb{A}^\kappa$ is an example of $\mathfrak{A}_c^\kappa$. The functional values of approximable maps can be convex, contractible, decomposable, or $\infty$-proximally connected whenever the domains of the maps are convex subsets of a locally convex t.v.s. See [BI].

We list some examples of intermediate spaces of a composite in $\mathfrak{A}_c(P,P)$:

(1) Any topological space for $\mathbb{C}$.
(2) Any subset of a t.v.s. for $\mathbb{K}$ [L1,B].
(3) Any metric space for $\mathbb{V}$ [Po].
(4) Any Hausdorff topological space for $\mathbb{V}$ [GG, Gr].
(5) Any subset of a t.v.s. for $\mathbb{A}$ [BI].

## 2. Coincidence and fixed point theorems

The notions of convex spaces and $H$-spaces were extended by the author as follows [PK1,3]:

A *generalized convex space* or a *G-convex space* $(X,D;\Gamma)$ consists of a topological space $X$, a nonempty subset $D$ of $X$, and a map $\Gamma : \langle D\rangle \multimap X$ such that

(1) for each $A, B \in \langle D\rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and
(2) for each $A \in \langle D\rangle$ with $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Note that $\langle D\rangle$ denotes the set of all nonempty finite subsets of $D$, $|A|$ the cardinality of $A$, $\Delta_n$ the standard $n$-simplex, and $\Delta_J$ the face of $\Delta_n$ corresponding to $J \in \langle A\rangle$. We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D\rangle$.

For an $(X, D; \Gamma)$, a subset $C$ of $X$ is said to be *G-convex* if for each $A \in \langle D\rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

We begin with the following coincidence theorem [PK3, Theorem 1].

**Theorem 2.1.** *Let $(X, D; \Gamma)$ be a G-convex space, $Y$ a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $x \in D$, $Sx$ is compactly open in $Y$;*
(2) *for each $y \in F(X)$, $M \in \langle S^- y\rangle$ implies $\Gamma_M \subset T^- y$;*
(3) *there exists a nonempty compact subset $K$ of $Y$ such that $\overline{F(X)} \cap K \subset S(D)$; and*
(4) *either*
   (i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D\rangle$; or*
   (ii) *for each $N \in \langle D\rangle$, there exists a compact G-convex subset $L_N$ of $X$ containing $N$ such that $F(L_N)\backslash K \subset S(L_N \cap D)$.*

*Then there exists an $\overline{x} \in X$ such that $F\overline{x} \cap T\overline{x} \neq \emptyset$.*

It is assumed in Theorem 2.1 that $Y$ is an intermediate space for $\mathfrak{A}$.

In [P2,3], particular forms of Theorem 2.1 are applied to obtain basic results in the KKM theory.

The following is due to the author [P2,3, PK3] by using Theorem 2.1:

**Theorem 2.2.** *Let $X$ be a nonempty convex subset of a locally convex t.v.s. $E$ and $T \in \mathfrak{A}_c^\sigma(X, X)$ a compact map. Then $T$ has a fixed point $x_0 \in X$; that is, $x_0 \in Tx_0$.*

It is not known yet whether Theorem 2.2 holds for $\mathfrak{A}_c^\kappa$ instead of $\mathfrak{A}_c^\sigma$. However, a partial generalization of Theorem 2.2 will be given in Theorem 3.3.

For the definition of an approximate neighborhood extension space for compact spaces (simply, ANES (compact)), see [BD2].

From [BD2, Proposition 2.2 and Theorem 2.3], we have the following:

**Theorem 2.3.** *Let $X$ be the Hilbert cube $I^\infty$ or any Tychonoff cube. Then any $F \in \mathfrak{A}_c(X, X)$ has a fixed point.*

**Theorem 2.4.** *If $X$ is an ANES (compact) and $F \in \mathfrak{A}_c(X, X)$ is compact, then $F$ has a fixed point.*

## 3. Fixed points of condensing maps

Let $E$ be a t.v.s. and $C$ a lattice with a least element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on $E$ provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if $X$ is relatively compact;

(2) $\Phi(\overline{\text{co}}\, X) = \Phi(X)$, where $\overline{\text{co}}$ denotes the convex closure of $X$; and

(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

The above notion is a generalization of the set-measure $\gamma$ and the ball-measure $\chi$ of noncompactness defined either in terms of a family of seminorms or a norm. For details, see [PF1,2].

From now on, *let $E$ have a measure of non-compactness* $\Phi$. For $D \subset E$, a map $T : D \multimap E$ is said to be $\Phi$-*condensing* provided that if $X \subset D$ and $\Phi(X) \leq \Phi(T(X))$, then $X$ is relatively compact; that is, $\Phi(X) = 0$.

Every map defined on a compact set and every compact map is $\Phi$-condensing.

In this section, we need the following consequence of Theorem 2.1 in [P1]:

**Theorem 3.1.** *Let $K$ be a nonempty compact convex subset of a t.v.s. $E$ on which $E^*$ separates points. Then any map $T \in \mathfrak{A}_c^\kappa(K, K)$ has a fixed point.*

The following is proved for a locally convex t.v.s. by Mehta, Tan, and Yuan [MTY, Lemma 1], but the proof works for any t.v.s.:

**Lemma.** *Let $D$ be a nonempty closed convex subset of a t.v.s. $E$ and $T : D \multimap D$ a $\Phi$-condensing map. Then there exists a nonempty compact convex subset $K$ of $D$ such that $T(K) \subset K$.*

From Theorem 3.1 and Lemma, we obtain the following:

**Theorem 3.2.** *Let $D$ be a nonempty closed convex subset of a t.v.s. $E$ on which $E^*$ separates points. Then any $\Phi$-condensing map $T \in \mathfrak{A}_c^\kappa(D, D)$ has a fixed point.*

*Proof.* By Lemma, there exists a nonempty compact convex subset $K$ of $D$ such that $T(K) \subset K$. Note that $T|_K \in \mathfrak{A}_c^\kappa(K, K)$ has a fixed point by Theorem 3.1.

Since every compact map is $\Phi$-condensing, from Theorem 3.2, we have the following partial generalization of Theorem 2.2:

**Theorem 3.3.** *Let $D$ be a nonempty closed convex subset of a t.v.s. $E$ on which $E^*$ separates points. Then any compact map $T \in \mathfrak{A}_c^\kappa(D, D)$ has a fixed point.*

From Theorem 3.2, we can deduce the Leray-Schauder type principle for condensing admissible maps as in [P13]:

Let $C, D$ be subsets of a t.v.s. $E$, $T \in \mathfrak{A}_c(C, D)$, and $\mathcal{M}$ be the class of nonempty compact subsets of $D$ consisting of the functional values of maps in $\mathfrak{A}$. We say that $F$ satisfies the *Schöneberg condition* if

(Sö) $tM \in \mathcal{M}$ for $t \in [0, 1]$ and $M \in \mathcal{M}$

holds [Sö]. For example, $\mathcal{M}$ cna be the class of convex sets for $\mathfrak{A} = \mathbb{K}$, acyclic sets for $\mathfrak{A} = \mathbb{V}$, $R_\delta$ sets $\{X = \cap X_i : X_{i+1} \subset X_i,\ X_i \in \text{AR compact},\ i \in \mathbf{N}\}$ for $\mathfrak{A} = \mathbb{M}$, and many others.

Following the method of Schöneberg [Sö, Theorem] or [P13, Theorem 4], from Theorem 3.2, we can deduce

**Theorem 3.4.** *Let $D$ be a closed convex subset of a t.v.s. $E$ on which $E^*$ separates points, $0 \in D$, and $U \subset D$ a neighborhood of $0$ (in $D$). Let $H \in \mathfrak{A}([0,1] \times \overline{U},\ D)$ satisfy condition (Sö) and*

(1) $x \notin H(t,x)$ *for* $t \in [0,1)$ *and* $x \in \mathrm{Bd}_D U$*;*

(2) $H(1,x) \cap \{\lambda x : \lambda > 1\} = \emptyset$ *for all* $x \in \mathrm{Bd}_D U$*; and*

(3) $X \subset \overline{U}$ *and* $\Phi(X) \leq \Phi(H([0,1] \times X))$ *imply that* $X$ *is relatively compact.*

*Then there exist a $\Phi$-condensing map $G \in \mathfrak{A}(D,D)$ and an $x \in D$ such that $x \in Gx$ or, equivalently, $x \in U$ and $x \in H(0,x)$.*

## 4. Variations of the Leray-Schauder principle

In [M], the author obtained generalizations of the Leray-Schauder principle, and applied them to surjectivity results for $A$-proper multifunctions and others. However, his generalizations seem to be incorrectly stated and not general enough.

In this final section of the present paper, we prove correct and generalized forms of the Leray-Schauder theorems in [M]. We use Theorem 3.4, whose proof does not depend on index theory or a retraction argument.

**Theorem 4.1.** *Let $D$ be a closed convex subset of a t.v.s. $E$ on which $E^*$ separates points, $0 \in D$, and $U \subset D$ a neighborhood of $0$ (in $D$). Let $\mu \geq 1$ and $H \in \mathfrak{A}([0,1] \times \overline{U},\ D)$ satisfy condition (Sö) and*

(1) $x \notin H(t,x)$ *for* $t \in [0,1)$ *and* $x \in \mathrm{Bd}_D U$*;*

(2) *if* $\lambda x \in H(1,x)$ *for some* $x \in \mathrm{Bd}_D U$*, then* $\lambda \leq \mu$*; and*

(3) $X \subset \overline{U}$ *and* $\Phi(X) \leq \Phi(H([0,1] \times X))$ *imply that* $X$ *is relatively compact.*

*Then there exist a $\Phi$-condensing map $G \in \mathfrak{A}(D,D)$ and an $x \in D$ such that $x \in Gx$ or, equivalently, $x \in \overline{U}$ and $\mu x \in H(0,x)$.*

*Proof.* Define the map $H' \in \mathfrak{A}([0,1] \times \overline{U},\ D)$ by $H'(t,x) := (1/\mu)H(t,x)$. This is well-defined since $0 \in D$, $H(t,x) \subset D$, $D$ is convex, and $\mathfrak{A}$ satisfies (Sö). Then, by (1), $x \notin H'(t,x)$ for $t \in [0,1]$ and $x \in \mathrm{Bd}_D U$. Moreover, by (2), $\lambda x \notin H(1,x)$

for $x \in \mathrm{Bd}_D U$ and $\lambda > \mu$, whence $(\lambda/\mu)x \notin H'(1,x)$ for $\lambda/\mu > 1$. This implies $H'(1,x) \cap \{sx : s > 1\} = \emptyset$ for all $x \in \mathrm{Bd}_D U$. Further, for any $X \subset \mathrm{Cl}_D U$, we have

$$H'([0,1] \times X) \subset \overline{\mathrm{co}}\{H(\{0\} \cup ([0,1] \times X)\},$$

whence $\Phi(H'([0,1] \times X)) < \Phi(X)$ whenever $\Phi(X) \neq 0$. Therefore all of the requirements of Theorem 3.4 with $H = H'$ are satisfied, and the conclusion follows.

Note that for a Fréchet space $E$, a cone $D$, and $\mathfrak{A} = \mathbb{K}$, Theorem 4.1 reduces to a correct form of [M, Theorem 1.1]. Similarly [M, Corollary 1.1] can be improved as follows:

**Theorem 4.2.** *Let $K$ be a closed cone of a t.v.s. $E$ on which $E^*$ separates points, $U \subset K$ a neighborhood of $0$ (in $D$). Let $\mu \in (0,1]$ and $H \in \mathbb{K}([0,1] \times \overline{U}, D)$ satisfy the following:*

(1) *$x \notin \mu H(t,x)$ for $t \in [0,1)$ and $x \in \mathrm{Bd}_D U$;*

(2) *$\lambda x \notin H(1,x)$ whenever $x \in \mathrm{Bd}_K U$ and $\lambda > 1/\mu$; and*

(3) *$X \subset \overline{U}$ and $\Phi(X) \leq \Phi(H([0,1] \times X))$ imply that $X$ is relatively compact.*

*Then there exists an $x \in \overline{U}$ such that $x \in \mu H(0,x)$.*

*Proof.* Define the map $H' \in \mathbb{K}([0,1] \times \overline{U}, K)$ by $H'(t,x) := (1/\mu)H(t,x)$, where $1/\mu \geq 1$. This is well-defined since $H(t,x) \subset K$ and $K$ is a cone. Now we can apply Theorem 3.4 or 4.1 and obtain an $x \in \overline{U}$ such that $(1/\mu)x \in H(0,x)$. This completes our proof.

From Theorems 4.1 and 4.2, we obtain the following Leray-Schauder type alternatives:

**Theorem 4.3.** *Let $D$ be a closed convex subset of a t.v.s. on which $E^*$ separates points, $0 \in D$, and $U \subset D$ a neighborhood of $0$ (in $D$), $T \in \mathfrak{A}(\overline{U}, D)$ satisfy condition (Sö), and $\mu \geq 1$. If $T$ is $\Phi$-condensing, then either*

(1) *there exists an $x \in U$ such that $\mu x \in Tx$; or*

(2) *there exists an $x \in \mathrm{Bd}_D U$ such that $\lambda x \in Tx$ for some $\lambda > \mu$.*

*Proof.* We use Theorem 4.1 with $H(t,x) = Tx$ for $t \in [0,1]$ and $x \in \overline{U}$. Suppose that (2) does not hold. If there exists an $x_0 \in \mathrm{Bd}_D U$ such that $\mu x_0 \in Tx_0$, then we have done. If there is no $x_0 \in \mathrm{Bd}_D U$ satisfying $\mu x_0 \in Tx_0$, then all of the requirements of Theorem 4.2 are satisfied. Therefore, there exists an $x_0 \in \overline{U}$ such that $\mu x_0 \in Tx_0$.

**Theorem 4.4.** *Let $D$ be a closed subset of a t.v.s. on which $E^*$ separates points, $0 \in \mathrm{Int}\, D$, $T \in \mathfrak{A}(D, E)$ a $\Phi$-condensing map satisfying (Sö), and $\mu \geq 1$. Then either*

(1) *there exists an $x \in D$ such that $\mu x \in Tx$; or*

(2) *there exists an $x \in \mathrm{Bd}\, D$ such that $\lambda x \in Tx$ for some $\lambda > \mu$.*

*Proof.* Use Theorem 4.3 with $(E, \mathrm{Int}\, D)$ instead of $(D, U)$.

Note that Theorems 4.3 and 4.4 generalize some results in [Gr] and [FM].

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