GENERALIZED QUASI-VARIATIONAL INEQUALITIES

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Abstract. Existence theorems for two variable functions or generalized quasi-variational inequalities on topological vector spaces are deduced from general fixed point theorems due to the first author [P2-4]. Our new results include a number of known variational or variational-like inequalities and other known theorems.

In this paper, from general fixed point theorems recently due to the first author [P2-4], we deduce existence theorems for two variable functions on convex subsets of a Hausdorff topological vector space. Our new theorems can be considered as quasi-variational inequalities. We show that our new results include a number of known variational or variational-like inequalities and other known theorems.

A *multifunction* or *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) = \{x \in X : y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x) : x \in A\}$. A multifunction $F : X \multimap Y$ is *compact* provided $F(X)$ is contained in a compact subset of Y. Given two maps $F : X \multimap Y$ and $G : Y \multimap Z$, the *composite* $GF : X \multimap Z$ is defined by $(GF)(x) = G(F(x))$ for $x \in X$.

For topological spaces X and Y, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X. Note that composites

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of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact. Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

In a topological vector space any convex hulls of its finite subsets will be called *polytopes*.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values) [Gr], the acyclic maps $\mathbb{V}$ (with acyclic values), the O'Neill maps $\mathbb{N}$ (with values of one or m acyclic components, where m is fixed) [Gr], the approachable maps $\mathbb{A}$ in a t.v.s. [BD], admissible maps in the sense of Górniewicz [G], permissible maps of Dzedzej [D], and others.

We introduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ-compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K, Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K, Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

Any class $\mathfrak{A}_c^\kappa$ will be called *admissible*. For details, see [P2, PK].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [L] and $\mathbb{V}_c^\sigma$ due to Park, Singh, and Watson [PSW]. Note that $\mathbb{K}_c^\sigma$ contains classes $\mathbb{K}$, Fan-Browder type maps, and $\mathbb{T}$ in [L]. Moreover, the class of approximable maps due to Ben-El-Mechaiekh and Idzik [BI] is an example of $\mathfrak{A}_c^\kappa$. The functional values of approximable maps can be convex, contractible, decomposable, or ∞-proximally connected whenever the domains of the maps are convex subsets of a locally convex Hausdorff topological vector space [BI].

For a class $\mathfrak{A}(X, Y)$, the set of functional values in 2^Y will be denoted by $\mathfrak{A}(Y)$. For example, $\mathbb{C}(Y) = Y$, $\mathbb{K}(Y)$ is the set of nonempty compact convex sets, $\mathbb{V}(Y)$ compact acyclic sets, and so on.

The following fixed point theorem is recently due to the first author [P2,3].

Theorem 1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E, and $F \in \mathfrak{A}_c^\sigma(X, X)$. If F is compact, then there exists an $x_0 \in X$ such that $x_0 \in F(x_0)$.*

Note that Theorem 1 contains a large number of historically well-known fixed point theorems. One of them is the Himmelberg fixed point theorem [H], which is Theorem 1 for $\mathbb{K}$ replacing $\mathfrak{A}_c^\sigma$. Note also that Theorem 1 is equivalent to its particular form for $\mathfrak{A}_c$ replacing $\mathfrak{A}_c^\sigma$.

Recall that an extended real function $f : X \to \overline{\mathbf{R}}$ on a topological space X is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X : f(x) > r\}$ [resp., $\{x \in X : f(x) < r\}$] is open for each $r \in \overline{\mathbf{R}}$.

From Theorem 1 we have the following form of a quasi-variational inequality:

Theorem 2. *Let X be a convex subset of a locally convex Hausdorff topological vector space E, Y a nonempty compact subset of X, and $f : X \times Y \to \mathbf{R}$ an u.s.c. function. Let $S : X \multimap Y$ be an u.s.c. map with compact values. Suppose that*

(1) *the function M on X defined by*

$$M(x) = \sup_{y \in S(x)} f(x, y) \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$\{y \in S(x) : f(x, y) = M(x)\}$$

belongs to $\mathfrak{A}(Y)$ for some admissible class $\mathfrak{A}$.

Then there exists an $\hat{x} \in Y$ such that

$$\hat{x} \in S(\hat{x}) \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Proof. Note that the marginal function M in (1) is actually continuous by a well-known theorem of Berge [Be]. See also [AE]. Define a map $T : X \multimap Y$ by

$$T(x) = \{y \in S(x) : f(x, y) = M(x)\}$$

for $x \in X$. Note that each $T(x)$ is nonempty and belongs to $\mathfrak{A}(Y)$. Moreover, the graph $Gr(T)$ is closed in $X \times Y$. In fact, let $(x_\alpha, y_\alpha) \in Gr(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f(x,y) &\geq \overline{\lim_\alpha}\, f(x_\alpha, y_\alpha) = \overline{\lim_\alpha}\, M(x_\alpha) \\ &\geq \underline{\lim_\alpha}\, M(x_\alpha) \geq M(x) \end{aligned}$$

and, since $Gr(S)$ is closed in $X \times Y$, $y_\alpha \in S(x_\alpha)$ implies $y \in S(x)$. Hence $(x, y) \in Gr(T)$. This also shows that each $T(x)$ is closed and hence compact in Y. Therefore, $T \in \mathfrak{A}(X, Y)$ and hence, by Theorem 1, T has a fixed point $\hat{x} \in Y$; that is, $\hat{x} \in S(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. This completes our proof.

Remarks. 1. Note that Theorems 1 and 2 are equivalent. In fact, if $f(x,y) \equiv 0$ for all $x, y \in X$, then Theorem 2 reduces to Theorem 1.

2. If S and f are continuous in Theorem 2, then condition (1) holds by the maximum theorem of Berge [Be]. See [AE].

3. If $\mathfrak{A}(Y)$ is the class of compact acyclic subsets of Y, then Theorem 2 reduces to the first author [P5, Theorem 1], which was shown to include results of Fan [F], Takahashi [T], Im and Kim [IK], and others.

4. We note that Theorem 2 includes a number of known variational or variational-like inequalities. For example, we have the following:

Corollary. (Hartman-Stampacchia [HS, Lemma 3.1]). *Let K be a compact convex set in $\mathbf{R}^n$ and $B : K \to \mathbf{R}^n$ a continuous map. Then there exists $u_0 \in K$ such that*

$$\langle B(u_0), v - u_0 \rangle \geq 0 \quad \textit{for} \quad v \in K$$

where $\langle \cdot, \ \cdot \rangle$ denotes the scalar product in $\mathbf{R}^n$.

Proof. Let $X = Y = K$, $f(x,y) = \langle B(x), -y \rangle$, and $S(x) = K$ for $x, y \in K$, in Theorem 1. Then for each $x \in K$,

(1) holds automatically; and

(2) the set $\{y \in K : \langle B(x), -y \rangle = \sup_{y \in K} \langle B(x), -y \rangle\}$ is nonempty closed and convex.

Therefore, by Theorem 2 with $\mathfrak{A} = \mathbb{K}$, there exists a $u_0 \in K$ such that

$$\langle B(u_0), -u_0 \rangle = \sup_{v \in K} \langle B(u_0), -v \rangle \geq \langle Bu_0, -v \rangle \quad \text{for} \quad v \in K;$$

that is,

$$\langle B(u_0), v - u_0 \rangle \geq 0 \quad \text{for} \quad v \in K.$$

Note that the inequality holds for all $v \in \overline{I}_K(u_0)$, the closure of the inward set $I_K(u_0) = \{u_0 + r(x - u_0) \in E : x \in K,\ r > 0\}$.

Similarly, Theorem 2 gives simple proofs of the variational or variational-like inequalities due to Browder [B1, Theorem 3; B2, Theorem 2], Lions and Stampacchia [LS] (see Mosco [M, p.94]), Karamardian [K, Lemmas 3.1 and 3.2], Juberg and Karamardian [JK, Lemma], Park [P1, Corollary 1.3], Parida, Sahoo, and Kumar [PSK, Theorem 3.1], Behera and Panda [BP, Theorem 2.2], Siddiqi, Khaliq, and Ansari [SKA, Theorem 3.2]. Note that these results are stated for locally convex spaces.

From Theorem 1, we also have the following equivalent form of a coincidence theorem:

Theorem 3. *Let X be a nonempty set, E a locally convex Hausdorff topological vector space, $T : X \multimap E$ a map such that $T(X)$ is a convex subset of E, and $P : X \multimap T(X)$ is a compact map such that $PT^- \in \mathfrak{A}_c^\sigma(T(X), T(X))$. Then T and P have a coincidence point $x_0 \in X$; that is, $T(x_0) \cap P(x_0) \neq \emptyset$.*

Proof. By Theorem 1, PT^- has a fixed point $y_0 \in T(X)$; that is, $y_0 \in PT^-(y_0)$. Hence $y_0 \in Px_0$ for some $x_0 \in T^-(y_0)$; whence we have $y_0 \in Px_0 \cap Tx_0$.

Remarks. 1. If X is a nonempty convex subset of E and T is the inclusion, then Theorem 3 reduces to Theorem 1.

2. Theorem 3 is motivated by S.-S. Chang [C, Theorem 5], which is a very particular form of Theorem 3 equivalent to Fan's fixed point theorem.

For non-locally convex spaces, the first author also obtained the following fixed point theorem [P4]:

Theorem 4. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E on which E^* separates points. Then any $F \in \mathfrak{A}_c^\kappa(X, X)$ has a point $x_0 \in X$ such that $x_0 \in F(x_0)$.*

Note that Theorem 4 also contains a large number of well-known fixed point theorems. See [P4].

From Theorem 4 we have the following form of a quasi-variational inequality:

Theorem 5. *Let X be a nonempty compact convex subset of a topological vector space E on which E^* separates points, and $f : X \times Y \to \mathbf{R}$ an u.s.c. function. Let $S : X \multimap Y$ be an u.s.c. map with compact values. Suppose that conditions* (1) *and* (2) *of Theorem 2 with $Y = X$ holds.*

Then there exists an $\hat{x} \in Y$ such that

$$\hat{x} \in S(\hat{x}) \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Remarks. 1. If $\mathfrak{A} = \mathbb{V}$, then Theorem 5 reduces to Park [P5, Theorem 5] where $X = Y$.

2. Using Theorem 5, we can state all of the variational or variational-like inequalities listed just after the Corollary for topological vector spaces E on which E^* separates points.

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