REMARKS ON FIXED POINTS OF GENERALIZED UPPER HEMICONTINUOUS MAPS

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Abstract. We give new fixed point theorems on a generalized upper hemicontinuous multifunction whose domain and range may have different topologies. These include known theorems appeared in almost 50 published works.

1. Introduction

In our previous works [P5,6], we unified, improved, and generalized a lot of fixed point theorems on Kakutani maps or acyclic maps defined on convex subsets of topological vector spaces. One of the main fixed point theorems in [P6] is concerned with generalized upper hemicontinuous maps whose domains and ranges may have different topologies. After the author completed the paper [P6], he came to know that there have appeared a number of results of this kind; for example, Roux and Singh [RS], Ding and Tan [DT1], and others.

In the present paper, we obtain some refined and generalized versions of main theorems in [P5,6] with slightly different proofs. We also show that some old or recent results of others are consequences of ours. Our results contain known theorems of Sehgal and Singh [SS], Roux and Singh [RS], Kim and Tan [KT], Ding and Tan [DT], and many others.

2. Preliminaries

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A *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$.

Let E be a Hausdorff topological vector space (t.v.s.) and E^* its topological dual. A multifunction or set-valued map (simply, map) $F : X \to 2^E \backslash \{\emptyset\}$ is said to be *upper hemicontinuous* (u.h.c.) if for each $h \in E^*$ and for any real α, the set $\{x \in X : \sup \operatorname{Re} h(Fx) < \alpha\}$ is open in X.

Let $cc(E)$ denote the set of nonempty closed convex subsets of E and $kc(E)$ the set of nonempty compact convex subsets of E. Bd, Int, and — denote the boundary, interior, and closure, resp., with respect to E.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

For $p \in \{\operatorname{Re} h : h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)| \ : \ u \in U, \ v \in V\}.$$

Recall that a real function $g : X \to \mathbb{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : gx > r\}$ [resp. $\{x \in X : gx < r\}$] is open for each $r \in \mathbb{R}$. If X is a convex set, then g is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : gx > r\}$ [resp. $\{x \in X : gx < r\}$] is convex for each $r \in \mathbb{R}$.

In this paper all topological spaces are assumed to be Hausdorff.

We use the following form of the existence theorem of maximizable quasiconcave functions on convex spaces due to Bellenger [B] and Park and Bae [PB].

Theorem 0. *Let X be a convex space and $\hat{X}$ the set of all u.s.c. quasiconcave real functions on X. Suppose that*

(0.1) *for each $x \in X$, Sx is a nonempty convex subset of $\hat{X}$;*

(0.2) *for each $g \in \hat{X}$, $S^{-1}g$ is compactly open in X; and*

(0.3) *there exists a c-compact set $L \subset X$ and a nonempty compact set $K \subset X$ such that for every $x \in X \backslash K$ and $g \in Sx$, $gx < \max g[x, L]$.*

Then there exist an $\overline{x} \in K$ and a $g \in S\overline{x}$ such that $g\overline{x} = \max g(X)$.

3. Main results

We begin with the following generalization of [P5, Corollary 3.1]:

Theorem 1. *Let X be a convex space, L a c-compact subset of X, K a nonempty subset of X, E a t.v.s. containing X as a subset, and $F : X \to 2^E \backslash \{\emptyset\}$. Suppose that, for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$,*

(1.0) *$p|_X$ is continuous on X;*

(1.1) *$X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$ is compactly closed in X;*

(1.2) *$x \in K$ and $p(x) = \max p(X)$ implies $x \in X_p$; and*

(1.3) *$x \in X \backslash K$ and $p(x) = \max p[x, L]$ implies $x \in X_p$.*

Then there exists an $x \in \bigcap \{X_p : p \in \{\mathrm{Re}\, h : h \in E^*\}\}$.

Proof. Note that $\{(\mathrm{Re}\, h)|_X : h \in E^*\} \subset \hat{X}$ by (1.0). For each $x \in X$, define

$$Sx = \{p|_X : p \in \{\mathrm{Re}\, h : h \in E^*\} \text{ and } \sup p(Fx) < p(x)\}.$$

Then Sx is a convex subset of $\hat{X}$. Suppose that $Sx \neq \emptyset$ for each $x \in X$; that is, for each $x \in X$, there exists a $p \in \{\mathrm{Re}\, h : h \in E^*\}$ such that $x \notin X_p$. Note that, for each $g \in \hat{X}$,

$$S^{-1}g = \{x \in X : \sup p(Fx) < p(x)\} = X \backslash X_p$$

if $g = p|_X$ for some $p \in \{\mathrm{Re}\, h : h \in E^*\}$ and

$$S^{-1}g = \emptyset \quad \text{if} \quad g \notin \{(\mathrm{Re}\, h)|_X : h \in E^*\}.$$

Then $S^{-1}g$ is compactly open in X for each $g \in \hat{X}$ by (1.1). Therefore, (0.1) and (0.2) are satisfied. Further, (1.3) implies (0.3). In fact, for every $x \in X \backslash K$ and $p \in \{\mathrm{Re}\, h : h \in E^*\}$ satisfying $\sup p(Fx) < p(x)$, we have $x \notin X_p$. Therefore, $p(x) < \max p[x, L]$ by (1.3). Now, by applying Theorem 0, there exist an $\overline{x} \in K$ and an $h \in E^*$ such that $p = \mathrm{Re}\, h$, $p|_X \in S\overline{x}$ and $p(\overline{x}) = \max p(X)$. Note that $p|_X \in S\overline{x}$ implies $\overline{x} \notin X_p$. This contradicts (1.2).

Remarks. 1. As we noted in [P6], in Theorem 1, we do not require any concrete connection between topologies of X and E except

(1.0) $(\mathrm{Re}\, h)|_X \in \hat{X}$ (that is, $(\mathrm{Re}\, h)|_X$ is continuous on X) for all $h \in E^*$.

In order to assure the continuity of $(\mathrm{Re}\, h)|_X$ for all $h \in E^*$, it is sufficient to assume that

(i) as a convex space, X has any topology finer than the relative weak topology with respect to E, and

(ii) E has any topology finer than its weak topology.

This is why there have appeared fixed point theorems on maps whose domains and ranges have different topologies.

2. If F is u.h.c. on each nonempty compact subset C of X, then F satisfies the "continuity" condition (1.1) for all $p \in \{\mathrm{Re}\, h : h \in E^*\}$, but not conversely. See [P5]. Any map F satisfying (1.1) can be said to be *generalized u.h.c.*

3. The "boundary" condition (1.2) is actually same to the following:

(1.2)′ $x \in K$ and $p(x) = \max p(\overline{I}_X(x))$ implies $x \in X_p$.

In fact, $p(x) = \max p(X)$ is equivalent to $p(x) = \max p(\overline{I}_X(x))$.

4. The "coercivity" or "compactness" condition (1.3) is actually same to the following:

(1.3)′ $x \in X \backslash K$ and $p(x) = \max p(\overline{I}_L(x))$ implies $x \in X_p$.

In fact, $p(x) = \max p[x, L]$ is equivalent to $p(x) = \max p(\overline{I}_L(x))$. Note that if X itself is compact (that is, if $X = K$), then (1.3)′ holds trivially.

From Theorem 1, we have the following basic fixed point theorem:

Theorem 2. *Under the hypothesis of Theorem 1, further suppose that either*

(A) *E^* separates points of E and $F : X \to kc(E)$; or*

(B) *E is locally convex and $F : X \to cc(E)$.*

Then there exists an $x \in X$ such that $x \in Fx$.

Proof. By Theorem 1, there exists an $x \in \bigcap\{X_p : p \in \{\mathrm{Re}\, h : h \in E^*\}\}$. Suppose that $x \notin Fx$. Then under the assumptions (A) or (B), the standard separation theorems on a t.v.s. assure the existence of a $p \in \{\mathrm{Re}\, h : h \in E^*\}$ satisfying $\inf p(Fx) > p(x)$; that is, $x \notin X_p$, which is a contradiction.

Remarks. 1. Using the method in [P5], we can reformulate Theorem 2 to a coincidence theorem and an existence theorem for critical points or zeros of multifunctions.

2. Note that $x \in Fx$ if and only if $x \in \bigcap\{X_p : p \in \{\mathrm{Re}\, h : h \in E^*\}\}$. This is a useful information on the location of a fixed point.

From Theorem 2, we obtain the following more visualizable geometric form of a fixed point and surjectivity theorem, which generalizes [P5, Theorem 6] and refines [P6, Theorem 2]:

Theorem 3. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*

(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$,*

(1.0) *$p|_X$ is continuous on X;*

(3.1) *$X_p = \{x \in X : \inf p(Fx) \le p(x)\}$ is compactly closed in X;*

(3.2) *$d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \mathrm{Bd}\, X$; and*

(3.3) *$d_p(Fx, \overline{I}_L(x)) = 0$ for every $x \in X \backslash K$.*

Then there exists an $x \in X$ *such that* $x \in Fx$.

(II) *Suppose that for each* $p \in \{\mathrm{Re}\, h : h \in E^*\}$,

(1.0) $p|_X$ *is continuous on* X;

$(3.1)'$ $X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$ *is compactly closed in* X;

$(3.2)'$ $d_p(Fx, \overline{O}_X(x)) = 0$ *for every* $x \in K \cap \mathrm{Bd}\, X$; *and*

$(3.3)'$ $d_p(Fx, \overline{O}_L(x)) = 0$ *for every* $x \in X \backslash K$.

Then there exists an $x \in X$ *such that* $x \in Fx$. *Further, if* F *is u.h.c., then* $F(X) \supset X$.

Proof. Note that, since $I_X(x) = E$ for $x \in \mathrm{Int}\, X$, (3.2) is actually same to the following:

$(3.2)^*$ $d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K$.

In order to use Theorem 2, we first show that (3.2) $\Longrightarrow$ (1.2).

Let $x \in K$ such that $p(x) = \max p(X)$. Suppose that $\inf p(Fx) > p(x)$. Then for any $v \in Fx$, $u \in X$, $z = x + r(u - x) \in I_X(x)$, and $r > 0$, we have

$$|p(v - z)| = p(v - x) + rp(x - u) \geq p(v - x) = p(v) - p(x)$$

and hence

$$d_p(Fx, \overline{I}_X(x)) = d_p(Fx, I_X(x)) \geq \inf p(Fx) - p(x) > 0.$$

This contradicts $(3.2)^*$. Therefore, we should have $\inf p(Fx) \leq p(x)$ or $x \in X_p$. Hence, (1.2) holds.

Similarly, we can show that (3.3) $\Longrightarrow$ (1.3). Note that (3.1) for all p is the same to (1.1) = $(3.1)'$ for all p. Therefore, all of the requirements of Theorem 2 are satisfied. Now by Theorem 2, Case (I) follows.

For (II) consider $2x - Fx$ instead of Fx in (I) as in [P5], we can conclude that F has a fixed point. For the surjectivity result, let $y \in X$. Consider $x \mapsto Fx + x - y$ instead of Fx and $[y, L]$ instead of L in Case (II). Then there exists an $x \in X$ such that $x \in Fx + x - y$; that is, $y \in Fx$. This completes our proof.

Remarks. 1. (3.1) and $(3.1)'$ are actually same.

2. Note that the map $x \mapsto Fx + x - y$ in the proof of Case (II) is u.h.c.

3. Note that if K is a weakly compact convex subset of a t.v.s. (E, τ) on which E^* separates points, then a continuous map $f : (K, \tau) \to (K, \tau)$ may have no fixed point. See Kakutani [K, Theorem 1]. In this case, $K_p = \{x \in K : p(fx) \le p(x)\}$ in Theorem 3(I) may not be closed for some $p \in \{\operatorname{Re} h : h \in E^*\}$.

We give some of the simplest examples of Theorem 3.

Examples. 1. [P2, Example 1]: Let $X = K = [0, 1]$ in $E = \mathbb{R}$, $fx = x$ for $x \in X \backslash (1/3, 2/3)$, and $fx = 1$ for $x \in (1/3, 2/3)$. Then the set $\{x \in X : p(x) \le p(fx)\}$ is closed for all $p \in E^*$. Note that $f : X \to X$ is not continuous, but has a fixed point by Theorem 3(I).

2. [P2, Example 2]: Let $X = K = [0, 1]$ and $E = \mathbb{R}$. For a given $c \in (0, 1)$, let $f : X \to X$ be a function such that $fx > x$ for $x < c$ and $fx < x$ for $x > c$. Then c is the only fixed point of f if and only if the set $\{x \in X : p(x) \le p(fx)\}$ is closed for all $p \in E^*$.

3. Let $X = (0, 1]$, $K = L = [1/2, 1]$, $E = \mathbb{R}$, and $f : X \to X$ be given by $fx = (x + 1)/2$. Then $\{x \in X : p(x) \le p(fx)\}$ is closed for all $p \in E^*$. Note that

$$fx = \frac{x+1}{2} \in \overline{I}_L(x) = [x, \infty) \quad \text{for all} \quad x \in (0, \frac{1}{2}) = X \backslash K.$$

Therefore, Theorem 3(I) works.

4. Particular results

(1) A particular form of Theorem 3 for the real case is given as [P5, Theorem 6], which unifies, improves, and generalizes historically well-known fixed point theorems published in nearly 40 papers. See the diagram in [P5, p.205].

Now we add some more known consequences of Theorem 3 as follows:

(2) Knaster, Kuratowski, and Mazurkiewicz [KKM, p.136]: If $f : B^n \to \mathbb{R}^n$ is a continuous map such that f maps $S^{n-1} = \operatorname{Bd} B^n$ back into B^n, then f has a fixed point.

This is the origin of the so-called Rothe boundary condition.

(3) Sehgal and Singh [SS, Corollary 2]: Let K be a convex and weakly compact subset of a real locally convex t.v.s. E and $f : K \to E$ a strongly continuous map such that $f(\operatorname{Bd} K) \subset K$. Then f has a fixed point.

Note that f satisfies (3.1).

(4) Deimling [D, p.93]: Let X be a nonempty closed bounded convex subset of a reflexive Banach space E, and $f : X \to X$ a weakly sequentially continuous map. Then f has a fixed point.

Equip E with the weak topology.

(5) Arino, Gautier, and Penot [AGP, Theorem 1]: Let X be a nonempty weakly compact convex subset of a metrizable locally convex t.v.s. E, and $f : X \to X$ a weakly sequentially continuous. Then f has a fixed point.

Note that f is weakly continuous.

(6) Roux and Singh [RS, Theorem 5]: Let (E, τ) be a t.v.s. on which E^* separates points, w the weak topology of E, K a nonempty τ-compact convex subset of E, and $f : (K, \tau) \to (E, w)$ a continuous inward map. Then f has a fixed point.

Here, inward means $fx \in I_X(x)$ for all $x \in K$.

(7) Roux and Singh [RS, Theorem 6]: Let (E, τ) be a t.v.s. on which E^* separates points, w the weak topology of E, K a nonempty w-compact convex subset of E, and $f : (K, w) \to (E, \tau)$ a continuous inward map. Then f has a fixed point.

This contains some results in Sehgal, Singh, and Whitfield [SSW].

(8) Park [P2, Theorem]: Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points, and $f : X \to E$ a weakly inward [outward] map such that

$$\{x \in X : \operatorname{Re} h(x) < \operatorname{Re} h(fx)\}$$

is open for all $h \in E^*$. Then f has a fixed point.

Here, weakly inward means $fx \in \overline{I}_X(x)$ for all $x \in X$.

(9) Kim and Tan [KT, Theorem 2]: Let X be a nonempty paracompact bounded convex subset of a locally convex t.v.s. E, K a nonempty compact subset of X, and $F : X \to cc(E)$ an u.h.c. map satisfying the following:

(a) for each $x \in X$, $Fx \cap I_X(x) \neq \emptyset$; and

(b) for each $x \in X \backslash K$, $y \in X$ and $h \in E^*$, if $\operatorname{Re} h(y) > \inf \operatorname{Re} h(Fy)$, then $\operatorname{Re} h(y) \leq \operatorname{Re} h(x)$.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in F\hat{x}$.

Choose a point $y \in X$ and let $L = \{y\}$. If we replace (b) by

(b)′ for each $x \in X \backslash K$ and $h \in E^*$, $\operatorname{Re} h(x) < \inf \operatorname{Re} h(Fx)$ implies $\operatorname{Re} h(y) > \operatorname{Re} h(x)$.

Then (b)′ implies (1.3), which is equivalent to (3.3). See Jiang [J]. Therefore, in this case, the result follows from Theorem 3(B) for Case (I).

Actually, Kim and Tan based their argument on [KT, Corollary 2], which is quite different from [P3, Theorem 1].

(10) Kim and Tan [KT, Theorem 4]: Let X be a nonempty convex subset of a normed vector space E, K a nonempty compact subset of X, and $F : X \to cc(E)$ an u.h.c. map satisfying (a) and (b) in (9). Then there exists an $\hat{x} \in X$ such that $\hat{x} \in F\hat{x}$.

In this result, if we replace (b) by (b)′, then it also follows from Theorem 3(B) for Case (I). Further, note that in a normed vector space E, for any $A, B \in cc(E)$, $d(A, B) = 0$ if and only if $d_p(A, B) = 0$ for all $p \in \{\operatorname{Re} h : h \in E^*\}$, where d denotes the induced metric. See Jiang [J].

(11) Ding and Tan [DT2, Corollary 3]: Let X be a nonempty convex subset of a normed vector space E, and $G : X \to kc(E)$ continuous on each nonempty compact subset C of X. Suppose that there exist a nonempty compact convex subset L of X and a nonempty compact subset K of X such that

(i) for each $y \in K$, $Gy \cap \overline{I}_X(y) \neq \emptyset$ [resp. $Gy \cap \overline{O}_X(y) \neq \emptyset$];

(ii) for each $y \in X \backslash K$, $Gy \cap \overline{I}_L(y) \neq \emptyset$ [resp. $Gy \cap \overline{O}_L(y) \neq \emptyset$].

Then G has a fixed point.

This result contains Browder [Br, Corollaries 2 and 2′] and Shih and Tan [ST, Corollary 1].

Final Remark. The major particular forms of Theorem 3 can be adequately summarized by the following enlarged version of the diagrams given in [P1,4,5]. For the references which are not appeared in the end of this paper, see [P5].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex Hausdorff topological vector spaces, and IV for topological vector spaces having sufficiently many linear functionals. Moreover, f stands for single-valued maps and F for set-valued maps; and K stands for a nonempty compact convex subset of a space E, and X for a nonempty convex sub-

E	$f : K \longrightarrow K$		$F : K \longrightarrow 2^K$	
I	Brouwer	1912	Kakutani	1941
II	Schauder	1927, 1930	Bohnenblust and Karlin	1950
III	Tychonoff	1935	Fan Glicksberg	1952 1952
IV	Fan	1964	Granas and Liu	1986

	$f : K \longrightarrow E$		$F : K \longrightarrow 2^E$	
I	Bohl Knaster, Kuratowski and Mazurkiewicz	1904 1929		
II	Rothe	1938		
III	Halpern Fan Reich Sehgal and Singh	1965 1969 1972 1983	Browder Fan Glebov Halpern Cellina Reich Cornet Lasry and Robert Simons	1968 1969 1969 1970 1970 1972, 1978 1975 1975 1986
IV	Halpern and Bergman Kaczynski Roux and Singh Sehgal, Singh and Whitfield	1968 1983 1989 1990	Granas and Liu Park	1986 1988, 1991

			$F : X \longrightarrow 2^E$	
II			Ding and Tan	1992
III			Fan Shih and Tan Jiang	1984 1987, 1988 1988
IV			Park	1992, 1993

set of E satisfying certain coercivity conditions with respect to $F : X \to 2^E$ with certain boundary conditions.

In fact, Theorem 3 contains all of the fixed point theorems in the diagram. Note that, in the diagram, Bohl's theorem [**Bo**] in 1904 was well-known to be equivalent to Brouwer's theorem in 1912.

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