SOME EXISTENCE THEOREMS FOR TWO VARIABLE FUNCTIONS ON TOPOLOGICAL VECTOR SPACES

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Abstract. We obtain some existence theorems for two variable functions or quasi-variational inequalities on topological vector spaces as equivalent formulations of fixed point theorems due to the author [P2-4]. Our results extend works in [T], [IK], and [P1].

In this paper, from fixed point theorems recently due to the present author [P2-4], we deduce some existence theorems for two variable functions on topological vector spaces. Such existence theorems are shown to be equivalent to known fixed point theorems and extend previously known results of Park [P1], Takahashi [T], and Im and Kim [IK].

Let X and Y be sets. A *multifunction* $F : X \to 2^Y$ is a function from X into the class 2^Y of nonempty subsets of Y. For $A \subset X$, let $F(A) = \bigcup\{S(x) : x \in A\}$. For any $B \subset Y$, we denote

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\} \quad \text{and} \quad F^+(B) = \{x \in X : F(x) \subset B\}.$$

For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if $F^+(V)$ is open for each open set $V \subset Y$; *lower semicontinuous* (l.s.c.) if $F^-(V)$ is open for each open set $V \subset Y$; *continuous* if it is u.s.c. and l.s.c.; and *compact* if $F(X)$ is contained in a compact subset of Y. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

For topological spaces X and Y, we define

$f \in \mathbb{C}(X, Y) \Longleftrightarrow f$ is a (single-valued) continuous function.

$F \in \mathbb{K}(X, Y) \Longleftrightarrow F$ is a *Kakutani map*; that is, Y is a subset of a topological vector space and F is u.s.c. with compact convex values.

$F \in \mathbb{V}(X, Y) \Longleftrightarrow F$ is an *acyclic map*; that is, F is u.s.c. with compact acyclic values.

The following is due to the author [P2, Theorem 7(iii)] and a particular form of [P3, Theorem 3(iii)].

Theorem 1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E. Then any compact map $F \in \mathbb{V}(X, X)$ has a fixed point.*

Theorem 1 contains many particular cases and, for $\mathbb{K}$ instead of $\mathbb{V}$, it is known to be the Himmelberg fixed point theorem [H].

Recall that an extended real function $f : X \to \overline{\mathbb{R}}$ on a topological space X is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X : f(x) > r\}$ [resp., $\{x \in X : f(x) < r\}$ is open for each $r \in \overline{\mathbb{R}}$.

Theorem 1 has the following equivalent formulation of a form of quasi-variational inequality:

Theorem 2. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E, Y a nonempty compact subset of X, and $f : X \times Y \to \mathbb{R}$ an u.s.c. function. Let $S : X \to 2^Y$ be an u.s.c. multifunction with compact values. Suppose that*

(1) *the function M on X defined by*

$$M(x) = \sup_{y \in S(x)} f(x, y) \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$\{y \in S(x) : f(x, y) = M(x)\}$$

is acyclic.

Then there exists an $\hat{x} \in Y$ *such that*

$$\hat{x} \in S(\hat{x}) \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Proof of Theorem 2 using Theorem 1. Note that the marginal function M in (1) is actually continuous. See [AE] or [Be]. Define a multifunction $T : X \to 2^Y$ by

$$T(x) = \{y \in S(x) : f(x, y) = M(x)\}$$

for $x \in X$. Note that each Tx is nonempty and acyclic by (2). Moreover, the graph $\mathrm{Gr}(T)$ is closed in $X \times Y$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f(x, y) &\geq \overline{\lim_{\alpha}}\, f(x_\alpha, y_\alpha) = \overline{\lim_{\alpha}}\, M(x_\alpha) \\ &\geq \underline{\lim_{\alpha}}\, M(x_\alpha) \geq M(x) \end{aligned}$$

and, since $\mathrm{Gr}(S)$ is closed in $X \times Y$, $y_\alpha \in S(x_\alpha)$ implies $y \in S(x)$. Hence $(x, y) \in \mathrm{Gr}(T)$. This also shows that each $T(x)$ is closed and hence compact in Y. Therefore, $T \in \mathbb{V}(X, Y)$ and hence, by Theorem 1, T has a fixed point $\hat{x} \in Y$; that is, $\hat{x} \in S(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. This completes our proof.

Proof of Theorem 1 using Theorem 2. Put $S = F$, $Y = \overline{F(X)}$, and $f(x, y) = 0$ for $(x, y) \in X \times Y$. Then (1) and (2) hold automatically. Therefore, by Theorem 2, $F \in \mathbb{V}(X, Y)$ has a fixed point.

Remarks. 1. If $X = Y$, $S(x) = X$ for $x \in X$, and the set in (2) is convex, then Theorem 2 reduces to Takahashi [T, Theorem 4], which was applied to prove Fan's generalizations [F] of fixed point theorems of Schauder and Tychonoff. Fan's theorems are now called the best approximation theorems.

2. If S and f are continuous in Theorem 2, then (1) holds automatically. See Berge [Be]. In this case, Theorem 2 replacing acyclicity in (2) by convexity reduces to Im and Kim [IK, Theorem 1], which was used to generalize an existence theorem of Kaczynski and Zeidan [KZ] on non-compact infinite optimization problems.

3. In [IK], the authors gave an example showing that the lower semicontinuity of S is essential. However, their example actually shows that condition (1) is indispensable in Theorem 2.

4. If acyclicity in (2) is replaced by convexity, then Theorem 2 is equivalent to the Himmelberg theorem [H].

If M is replaced by a constant function in Theorem 2, then we have

Theorem 3. *Let X, E, Y, f, and S be the same as in Theorem 2. Let c be a real number such that*

(1) *$f(x, y) \leq c$ for every $(x, y) \in X \times Y$ with $y \in S(x)$; and*

(2) *for each $x \in X$, the set*

$$\{y \in S(x) : f(x, y) = c\}$$

is acyclic.

Then there exists an $\hat{x} \in \hat{Y}$ such that

$$\hat{x} \in S(\hat{x}) \quad \text{and} \quad f(\hat{x}, \hat{x}) = c.$$

Remark. If $X = Y$, $S(x) = X$ for $x \in X$, and the set in (2) is convex, then Theorem 3 reduces to Takahashi [T, Corollary 5].

If X itself is compact, then Theorem 2 holds for more general spaces than locally convex spaces. In order to show this, we need the following particular form of the author [P4, Theorem 5(vi)]:

Theorem 4. *Let X be a nonempty compact convex subset of a topological vector space E on which E^* separates points. Then any $F \in \mathbb{V}(X, X)$ has a fixed point.*

From Theorem 4, we have

Theorem 5. *Let X and E be the same as in Theorem 4. Let $f : X \times X \to \mathbb{R}$ be an u.s.c. function, and $S : X \to 2^X$ an u.s.c. multifunction with compact values such that conditions* (1) *and* (2) *of Theorem 2 hold. Then the conclusion of Theorem 2 holds.*

Proof. Just follow the proof of Theorem 2 using Theorem 4 instead of Theorem 1.

Remark. If E is locally convex, Hausdorff, and $S(x) = X$ for $x \in X$, then Theorem 4 reduces to Takahashi [T, Theorem 4].

For the definition of an *lc* space, see [B, P1]. Note that an ANR (metric) is an *lc* space and a finite union of compact convex subsets of a locally convex Hausdorff topological vector space is an *lc* space.

Theorem 6. *Let X be a compact acyclic lc space, $f : X \times X \to \mathbb{R}$ an u.s.c. function, and $S : X \to 2^X$ an u.s.c. multifunction with compact values. Suppose that conditions* (1) *and* (2) *of Theorem 2 hold. Then the conclusion of Theorem 2 holds.*

Proof. Just follow the proof of Theorem 2. Then the multifunction $T : X \to 2^X$ is acyclic-valued and has closed graph; that is, $T \in \mathbb{V}(X, X)$. Therefore, T has a fixed point by a well-known theorem of Begle [B].

Remarks. 1. If X is a compact convex subset of a locally convex Hausdorff topological vector space E and $S(x) = X$ for all $x \in X$, then Theorem 6 reduces to Takahashi [T, Theorem 4].

2. As for the case of Theorems 1 and 2, Theorem 6 is an equivalent formulations of Begle's theorem. Note that, for $S(x) = X$ for $x \in X$, Theorem 6 reduces to Park [P1, Theorem 1.3].

3. Finally note that Theorems 1 and 4 hold for admissible multifunctions of the form $\mathfrak{A}_c^\sigma$ and $\mathfrak{A}_c^\kappa$, respectively, in [P3, 4]. Therefore, some results in this paper can be improved by adopting more general classes of sets instead of acyclic sets.

References

[AE] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, New York, 1984.

[B] E. G. Begle, *A fixed point theorem*, Ann. Math. **51** (1950), 544–550.

[Be] C. Berge, *Espaces Topologique*, Dunod, Paris, 1959.

[F] Ky Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 234–240.

[H] C. J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.

[IK] S. M. Im and W. K. Kim, *An application of Himmelberg's fixed point theorem to non-compact optimization problems*, Bull. Inst. Math. Acad. Sinica **19** (1991), 1–5.

[KZ] T. Kaczynski and V. Zeidan, *An application of Ky Fan fixed point theorem to an optimization problem*, Nonlinear Anal. TMA **13** (1989), 259–261.

[P1] Sehie Park, *Fixed point theorems for compact acyclic metric spaces*, Proc. Amer. Math. Soc. **103** (1988), 946–950.

[P2] ———, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, "Fixed Point Theory and Applications" (K.-K. Tan, Ed.), World Sci., River Edge, NJ, 1992, pp.248–277.

[P3] ———, *Coincidences of composites of admissible u.s.c. maps and applications*, C.R. Math. Rep. Acad. Sci. Canada **15** (1993), 125–130.

[P4] ———, *Fixed point theory of multifunctions in topological vector spaces*, II, J. Korean Math. Soc. **30** (1993), 413–431.

[T] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings*, "Fixed Point Theory and Applications" (M.A. Théra and J.-B. Baillon, Eds.), Longman Sci. & Tech., Essex, 1991, pp.397–406.

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