

## COINCIDENCES OF COMPOSITES OF U.S.C. MAPS ON $H$ -SPACES AND APPLICATIONS

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### 1. Introduction

Applications of the classical Knaster - Kuratowski - Mazurkiewicz (simply, KKM) theorem and the fixed point theory of multifunctions defined on convex subsets of topological vector spaces have been greatly improved by adopting the concept of convex spaces due to Lassonde [L1]. In this direction, the first author [P5] found that certain coincidence theorems on a large class of composites of upper semicontinuous multifunctions imply many fundamental results in the KKM theory.

On the other hand, the concept of convex spaces was extended by Horvath [H1-5] to spaces having certain families of contractible subsets or  $H$ -spaces. In this direction, a number of authors extended important results on convex spaces to those on  $H$ -spaces. See Bardaro and Cepitelli [BC1-3], Ding *et al.* [Di, DKT1-2, DTr], Park [P1-3], Tarafdar [T], and H. Kim [K].

In the present paper, we extend the main coincidence theorem in [P5] to  $H$ -spaces and apply it to obtain a far-reaching generalization of the KKM theorem, fixed point or coincidence theorems for  $H$ -spaces, and other results. We also obtain open-valued versions of a KKM theorem and a coincidence theorem for  $H$ -spaces. Many of the main results in the above-mentioned papers are extended and unified. Especially, the main theorems of [Di] are improved and corrected.

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Received January 25, 1994.

1991 AMS Subject Classifications: Primary 54C65, 54H25, 52A07, 47H10, 55M20.

Key words and phrases: KKM theorem, multifunction, upper semicontinuous (u.s.c.), contractible, acyclic, convex space,  $D$ -convex, polytope,  $H$ -space,  $c$ -space,  $H$ -convex,  $H$ -subspace, Kakutani map, acyclic map, open-valued KKM theorem

Supported in part by the Basic Sciences Research Institute Program, Ministry of Education, 1992.

## 2. Preliminaries

For the terminology and notations, we follow mainly [P1-5].

A *multifunction* (simply, *map*)  $F : X \multimap Y$  is a function from a set  $X$  into the power set of a set  $Y$ . Let  $F(A) = \bigcup\{Fx : x \in A\}$  for  $A \subset X$  and  $F^{-1}y = \{x \in X : y \in Fx\}$  for  $y \in Y$ . As usual, the set  $\text{Gr}(F) = \{(x, y) : y \in Fx\}$  is called the *graph* of  $F$ .

For topological spaces  $X$  and  $Y$ , a map  $F : X \multimap Y$  is said to be *upper semicontinuous* (u.s.c.) if for each closed set  $B \subset Y$ ,  $F^{-1}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$  is closed in  $X$ ; and *compact* if  $F(X)$  is contained in a compact subset of  $Y$ .

A subset  $C$  of a topological space  $X$  is said to be *compactly closed* [resp. *open*] in  $X$  if for every compact set  $K \subset X$ , the set  $C \cap K$  is closed [resp. open] in  $K$ . A topological space  $X$  is said to be *contractible* if the identity map  $1_X$  of  $X$  is homotopic to a constant map; and *acyclic* if all of its reduced Čech homology groups over rationals vanish.

$\text{Int}$  and  $\overline{\phantom{x}}$  denote the interior and closure, resp.

For a set  $D$ , let  $\langle D \rangle$  denote the set of all nonempty finite subsets of  $D$ .

Let  $X$  be a set (in a vector space) and  $D$  a nonempty subset of  $X$ . Then  $(X, D)$  is called a *convex space* [P5] if for each  $N \in \langle D \rangle$ , its convex hull  $\text{co } N$  is contained in  $X$  and  $X$  has a topology that induces the Euclidean topology on such convex hulls. Such convex hull will be called a *polytope*. A subset  $A$  of  $X$  is said to be *D-convex* if, for any  $N \in \langle D \rangle$ ,  $N \subset A$  implies  $\text{co } N \subset A$ . If  $X = D$ , then  $X = (X, X)$  becomes a convex space in the sense of Lassonde [L1].

A triple  $(X, D; \Gamma)$  is called an *H-space* [P1,2] if  $X$  is a topological space,  $D$  a nonempty subset of  $X$ , and  $\Gamma = \{\Gamma_A\}$  a family of contractible subsets of  $X$  indexed by  $A \in \langle D \rangle$  such that  $\Gamma_A \subset \Gamma_B$  whenever  $A \subset B \in \langle D \rangle$ . (The triple is called a *c-space* in [H5] whenever  $X = D$ .) If  $X = D$ , we denote  $(X; \Gamma)$  instead of  $(X, X; \Gamma)$ . For an *H-space*  $(X; \Gamma)$  and any nonempty subset  $Y$  of  $X$ , we have an *H-space*  $(X, Y; \Gamma)$ .

Any convex space  $(X, D)$  is an *H-space*  $(X, D; \Gamma)$  by putting  $\Gamma_A = \text{co } A$  for  $A \in \langle D \rangle$ .

For an  $(X, D; \Gamma)$ , a subset  $C$  of  $X$  is said to be *H-convex* if for each  $A \in \langle D \rangle$ ,  $A \subset C$  implies  $\Gamma_A \subset C$ . A subset  $L$  of  $X$  is called an *H-subspace* of  $(X, D; \Gamma)$  if  $L \cap D \neq \emptyset$  and for every  $A \in \langle L \cap D \rangle$ ,

$\Gamma_A \cap L$  is contractible. This is equivalent to saying that the triple  $(L, L \cap D; \{\Gamma_A \cap L\})$  is an  $H$ -space.

Given a class  $\mathbf{X}$  of maps,  $\mathbf{X}(X, Y)$  denotes the set of all maps  $F : X \rightarrow Y$  belonging to  $\mathbf{X}$ , and  $\mathbf{X}_c$  the set of all finite composites of maps in  $\mathbf{X}$ .

For topological spaces  $X$  and  $Y$ , we define

$f \in \mathbf{C}(X, Y) \iff f$  is a (single-valued) continuous function.

$T \in \mathbf{K}(X, Y) \iff T$  is a *Kakutani map*; that is,  $Y$  is a convex space and  $T$  is u.s.c. with nonempty compact convex values.

$T \in \mathbf{V}(X, Y) \iff T$  is an *acyclic map*; that is,  $T$  is u.s.c. with compact acyclic values.

A class  $\mathfrak{A}$  of maps is one satisfying:

- (i)  $\mathfrak{A}$  contains the class  $\mathbf{C}$  of (single-valued) continuous functions;
- (ii) each  $F \in \mathfrak{A}_c$  is u.s.c. and compact-valued; and
- (iii) for any polytope  $P$ , each  $F \in \mathfrak{A}_c(P, P)$  has a fixed point.

Examples of  $\mathfrak{A}$  are  $\mathbf{C}, \mathbf{K}, \mathbf{V}$ , the Aronszajn maps  $\mathbf{M}$  (with  $R_\delta$  values) [Gr], the O'Neill maps  $\mathbf{N}$  (with values consisting of one or  $m$  acyclic components, where  $m$  is fixed) [Gr], the approachable maps in topological vector spaces [BD1,2], the admissible maps of Górniewicz [Go], the permissible maps of Dzedzej [Dz], and others.

Further, we define the following:

$T \in \mathfrak{A}_c^\sigma(X, Y) \iff$  for any  $\sigma$ -compact subset  $K$  of  $X$ , there is a  $\tilde{T} \in \mathfrak{A}_c(K, Y)$  such that  $\tilde{T}x \subset Tx$  for each  $x \in K$ .

$T \in \mathfrak{A}_c^\kappa(X, Y) \iff$  for any compact subset  $K$  of  $X$ , there is a  $\tilde{T} \in \mathfrak{A}_c(K, Y)$  as above.

The class  $\mathbf{K}_c^+$  due to Lassonde [L3] and  $\mathbf{V}_c^+$  due to Park, Singh, and Watson [PSW] belong to  $\mathfrak{A}_c^\sigma$ . Note that  $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$ . For details, see Park [P5] or [PK].

### 3. Coincidence theorems for $H$ -spaces

In this section, we give a basic coincidence theorem.

Let  $\Delta_n$  denote an  $n$ -simplex. We need the following:

LEMMA. *Let  $(X, D; \Gamma)$  be an  $H$ -space where  $D = \{x_0, x_1, \dots, x_n\} \in \langle X \rangle$ . Then there exists an  $f \in \mathbf{C}(\Delta_n, X)$  such that  $f(\Delta_J) \subset \Gamma_J$  for each  $J \in \langle D \rangle$ , where  $\Delta_J$  is the face of  $\Delta_n$  corresponding to  $J$ .*

Lemma is given implicitly by Horvath [H2, Théorème 1], [H3, Theorem 1] and explicitly by Ding and Tan [DT, Lemma 1] and Horvath [H4, Théorème 1], [H5, Theorem 1.1].

Our main result is the following general coincidence theorem related to the class  $\mathfrak{A}_c^\kappa$ :

**THEOREM 1.** *Let  $(X, D; \Gamma)$  be an  $H$ -space,  $Y$  a Hausdorff space,  $F \in \mathfrak{A}_c^\kappa(X, Y)$ , and  $K$  a nonempty compact subset of  $Y$ . Let  $S : D \multimap Y$  and  $T : X \multimap Y$  satisfy the following:*

- (1.1) *for each  $x \in D$ ,  $Sx$  is compactly open in  $Y$ ;*
- (1.2) *for each  $y \in F(X)$ ,  $M \in \langle S^-y \rangle$  implies  $\Gamma_M \subset T^-y$ ;*
- (1.3)  *$\overline{F(X)} \cap K \subset S(D)$ ; and*
- (1.4) *either*
  - (i)  *$Y \setminus K \subset S(M)$  for some  $M \in \langle D \rangle$ ; or*
  - (ii) *for each  $N \in \langle D \rangle$ , there exists a compact  $H$ -subspace  $L_N$  of  $X$  containing  $N$  such that  $F(L_N) \setminus K \subset S(L_N \cap D)$ .*

*Then  $T$  and  $F$  has a coincidence point  $\bar{x} \in X$ ; that is,  $T\bar{x} \cap F\bar{x} \neq \emptyset$ .*

*Proof.* Since  $\overline{F(X)} \cap K$  is compact and covered by compactly open sets  $Sx$  by (1.1) and (1.3), there exists an  $N \in \langle D \rangle$  such that  $\overline{F(X)} \cap K \subset S(N)$ .

Case (i). Since  $Y \setminus K \subset S(M)$  for some  $M \in \langle D \rangle$  by (i), we have  $\overline{F(X)} \subset S(A)$  where  $A = M \cup N = \{x_0, x_1, \dots, x_n\} \in \langle D \rangle$ . Then, by Lemma, there exists an  $f \in \mathcal{C}(\Delta_n, X)$  such that  $f(\Delta_n) \subset \Gamma_A$  and  $f(\Delta_J) \subset \Gamma_J$  for each  $J \in \langle A \rangle$ , where  $\Delta_n = \text{co}\{e_0, e_1, \dots, e_n\}$  and  $\Delta_J$  is the face of  $\Delta_n$  corresponding to  $J$ . Since  $f(\Delta_n)$  is compact in  $X$  and  $F \in \mathfrak{A}_c^\kappa(X, Y)$ , there exists a  $\tilde{F} \in \mathfrak{A}_c(f(\Delta_n), Y)$  such that  $\tilde{F}x \subset Fx$  for each  $x \in f(\Delta_n)$ . Then  $\tilde{F}f(\Delta_n)$  is compact in  $\overline{F(X)}$ , since it is the image of the compact set  $f(\Delta_n)$  under the compact valued u.s.c. map  $\tilde{F}$ . Let  $\{\lambda_i\}_{i=0}^n$  be the partition of unity subordinated to the cover  $\{Sx_i \cap \tilde{F}f(\Delta_n)\}_{i=0}^n$  of  $\tilde{F}f(\Delta_n)$ .

Define a continuous map  $p : \tilde{F}f(\Delta_n) \rightarrow \Delta_n$  by

$$py = \sum_{i=0}^n \lambda_i(y)e_i = \sum_{i \in N_y} \lambda_i(y)e_i \quad \text{for } y \in \tilde{F}f(\Delta_n),$$

where  $i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in Sx_i \iff x_i \in S^-y$ . By (1.2),

we have  $fp(y) \in f(\Delta_N) \subset \Gamma_N \subset T^{-1}y$  for each  $y \in \tilde{F}f(\Delta_n)$ ; that is,  $y \in (Tfp)y$ .

Since  $p\tilde{F}f \in \mathfrak{A}_c(\Delta_n, \Delta_n)$ ,  $p\tilde{F}f$  has a fixed point  $z \in \Delta_n$ ; that is,  $z \in (p\tilde{F}f)z$ . Put  $\bar{x} = fz$ . Since  $p^{-1}z \cap (\tilde{F}f)z = p^{-1}z \cap \tilde{F}\bar{x} \neq \emptyset$ , for any  $y \in p^{-1}z \cap \tilde{F}\bar{x}$ , we have  $y \in \tilde{F}f(\Delta_n)$ ,  $(fp)y = fz = \bar{x}$ , and  $y \in (Tfp)y = T\bar{x}$ . Therefore,  $p^{-1}z \cap \tilde{F}\bar{x} \subset T\bar{x}$  and hence  $T\bar{x} \cap \tilde{F}\bar{x} \subset T\bar{x} \cap F\bar{x} \neq \emptyset$ .

Case (ii). For an  $N \in \langle D \rangle$  such that  $\overline{F(X)} \cap K \subset S(N)$ , consider the set  $L_N$  in (1.4).

We claim that  $\tilde{F}(L_N) \subset S(L_N \cap D)$  for  $\tilde{F} \in \mathfrak{A}_c(L_N, Y)$  satisfying  $\tilde{F}x \subset Fx$  for each  $x \in L_N$ . In fact, note that

$$\tilde{F}(L_N) \cap K \subset F(X) \cap K \subset S(N) \subset S(L_N \cap D).$$

On the other hand,  $\tilde{F}(L_N) \setminus K \subset F(L_N) \setminus K \subset S(L_N \cap D)$  by (1.4). Therefore, we have  $\tilde{F}(L_N) \subset S(L_N \cap D)$ .

Note that  $\tilde{F}(L_N)$  is compact since it is the image of the compact set  $L_N$  under  $\tilde{F}$ . Therefore,  $\tilde{F}(L_N) \subset S(A)$  for some  $A = \{x_0, x_1, \dots, x_n\} \in \langle L_N \cap D \rangle$ . Then, by Lemma, there exists an  $f \in \mathbf{C}(\Delta_n, X)$  such that  $f(\Delta_n) \subset \Gamma_A \cap L_N = \Gamma'_A$  and  $f(\Delta_J) \subset \Gamma'_J$  for each  $J \in \langle A \rangle$ , where  $\Delta_n = \text{co}\{e_0, e_1, \dots, e_n\}$  and  $\Delta_J$  is the face of  $\Delta_n$  corresponding to  $J$ . Let  $\{\lambda_i\}_{i=0}^n$  be the partition of unity subordinated to the cover  $\{Sx_i \cap \tilde{F}f(\Delta_n)\}_{i=0}^n$  of  $\tilde{F}f(\Delta_n) \subset \tilde{F}(L_N)$ .

For the remainder of the proof, we can just follow that of Case (i). This completes our proof.

REMARKS. 1. Theorem 1 for Case (ii) is a correct and generalized form of Ding [Di, Theorem 3.3]. As in [Di], Theorem 1 can be applied to intersection theorems concerning sets with  $H$ -convex sections and the von Neumann type minimax theorems.

2. Note that, if  $F$  is single-valued, the Hausdorffness assumption is not necessary. See [P1, Theorem 6], [P2, Theorem 4].

PARTICULAR FORMS. 1. If  $(X, D; \Gamma)$  is a convex space with  $\Gamma_N = \text{co}N$  for each  $N \in \langle D \rangle$ , then Theorem 1 for Case (ii) includes Park [P5, Theorem 5], which was shown to be equivalent to a number of fundamental theorems in the KKM theory. For details, see [P5].

2. For  $H$ -spaces, Theorem 1 includes Horvath [H5, Theorem 4.2], Ding, Kim, and Tan [DKT1, Corollaries 3-5], Ding and Tan [DT, The-

orems 10-12 and Corollaries 2-4], Tarafdar [T, Theorem 2], Chen [Ch, Theorem 2], and Park [P1, Theorem 6], [P2, Theorem 4].

3. If  $F$  is compact in Theorem 1, by putting  $\overline{F(X)} = K$ , the coercivity condition (1.4) (ii) holds automatically. Therefore, we have the following:

**COROLLARY 1.** *Let  $(X, D; \Gamma)$  be an  $H$ -space,  $Y$  a Hausdorff space,  $F \in \mathfrak{A}_c^k(X, Y)$  a compact map, and  $S : D \rightarrow Y$ ,  $T : X \rightarrow Y$ . Suppose that*

- (1) *for each  $x \in D$ ,  $Sx$  is compactly open;*
- (2) *for each  $y \in F(X)$ ,  $M \in \langle S^{-1}y \rangle$  implies  $\Gamma_M \subset T^{-1}y$ ; and*
- (3)  *$\overline{F(X)} \subset S(D)$ .*

*Then  $F$  and  $T$  have a coincidence point  $\bar{x} \in X$ ; that is,  $F\bar{x} \cap T\bar{x} \neq \emptyset$ .*

**REMARK.** Condition (2) in Corollary 1 can be replaced by the following:

(2)' *for each  $x \in D$ ,  $Sx \subset Tx$  and, for each  $y \in F(X)$ ,  $T^{-1}y$  is  $H$ -convex.*

**PARTICULAR FORMS.** For  $H$ -spaces, Corollary 1 includes Horvath [H1, Théorème 4.1], [H2, Théorème 2 et Lemma 1], and [H3, Theorem 2'].

Moreover, from Corollary 1, we have

**COROLLARY 2.** *Let  $(X, D; \Gamma)$  be an  $H$ -space,  $Y$  a Hausdorff space,  $F \in \mathfrak{A}_c^k(X, Y)$  compact, and  $G : X \rightarrow Y$ . Suppose that*

- (1) *for each  $y \in F(X)$ ,  $G^{-1}y$  is  $H$ -convex; and*
- (2)  *$\{\text{Int } Gx : x \in D\}$  covers  $\overline{F(X)}$ .*

*Then  $F$  and  $G$  have a coincidence point.*

*Proof.* Put  $G = T$  and let  $S : D \rightarrow Y$  be defined by  $Sx = \text{Int } Gx$  for  $x \in D$  in Corollary 1.

**PARTICULAR FORMS.** If  $(X, D; \Gamma)$  is a convex space, then Corollary 2 reduces to Park [P5, Theorem 2]. For  $H$ -spaces, Corollary 2 includes Horvath [H4, Corollaire 6] and Ding and Tarafdar [DTr, Theorem 3.1].

4. The KKM theorems

In this section, we show that Theorem 1 is equivalent to the following KKM type theorem:

**THEOREM 2.** *Let  $(X, D; \Gamma)$  be an  $H$ -space,  $Y$  a Hausdorff space, and  $F \in \mathfrak{A}_c^*(X, Y)$ . Let  $G : D \multimap Y$  be a map such that*

- (2.1) *for each  $x \in D$ ,  $Gx$  is compactly closed in  $Y$ ;*
- (2.2) *for each  $N \in \langle D \rangle$ ,  $F(\Gamma_N) \subset G(N)$ ; and*
- (2.3) *there exists a nonempty compact subset  $K$  of  $Y$  such that either*
  - (i)  $\bigcap \{Gx : x \in M\} \subset K$  *for some  $M \in \langle D \rangle$ ; or*
  - (ii) *for each  $N \in \langle D \rangle$ , there exists a compact  $H$ -subspace  $L_N$  of  $X$  containing  $N$  such that*

$$F(L_N) \cap \bigcap \{Gx : x \in L_N \cap D\} \subset K.$$

Then we have

$$\overline{F(X)} \cap K \cap \bigcap \{Gx : x \in D\} \neq \emptyset.$$

*Proof.* Let  $S : D \multimap Y$ ,  $H : Y \multimap X$ , and  $T : X \multimap Y$  be defined by  $Sx = Y \setminus Gx$  for  $x \in D$ ,  $Hy = \bigcup \{\Gamma_M : M \in \langle S^{-y} \rangle\}$  for  $y \in Y$ , and  $Tx = H^{-}x$  for  $x \in X$ . Then (1.1) and (1.2) hold by (2.1) and the definitions of  $S$  and  $T$ . Suppose that  $\overline{F(X)} \cap K \cap \bigcap \{Gx : x \in D\} = \emptyset$ ; that is,  $\overline{F(X)} \cap K \subset S(D)$ , which is just (1.3). Note that (2.3) is equivalent to (1.4). Therefore, by Theorem 1,  $T$  and  $F$  have a coincidence point  $\bar{x} \in X$ ; that is,  $F\bar{x} \cap T\bar{x} \neq \emptyset$ . For  $y \in F\bar{x} \cap T\bar{x}$ , we have  $\bar{x} \in T^{-}y$  and hence, there exists an  $N \in \langle S^{-y} \rangle \subset \langle D \rangle$  such that  $\bar{x} \in \Gamma_N$ . Since  $y \in F\bar{x} \subset F(\Gamma_N) \subset G(N) = Y \setminus \bigcap \{Sx : x \in N\}$ ,  $y \notin Sx$  for some  $x \in N$ ; that is,  $x \notin S^{-}y$  and  $x \in N$ , which is a contradiction. This completes our proof.

Theorems 1 and 2 are equivalent:

*Proof of Theorem 1 using Theorem 2.* Let  $Gx = Y \setminus Sx$  for  $x \in D$ . Then (2.1) and (2.3) follow from (1.1) and (1.4), respectively. Moreover, from (1.3), we have

$$\overline{F(X)} \cap K \subset S(D) = \bigcup_{x \in D} (Y \setminus Gx) = Y \setminus \bigcap_{x \in D} Gx,$$

contrary to the conclusion of Theorem 2. Therefore,  $F$  and  $G$  does not satisfy (2.2) and hence, there exist an  $N \in \langle D \rangle$  and a  $y \in F(\Gamma_N) \setminus G(N)$ ; that is,  $y \notin Gx$  or  $y \in Sx$  for all  $x \in N$ . Since  $N \in \langle S^{-y} \rangle \subset \langle D \rangle$ , we have  $\Gamma_N \subset T^{-y}$  by (1.2). Since  $y \in F(\Gamma_N)$ , there exists an  $\bar{x} \in \Gamma_N$  such that  $y \in F\bar{x}$ . Note that  $\bar{x} \in \Gamma_N \subset T^{-y}$ ; that is,  $y \in T\bar{x}$ . This completes our proof.

**PARTICULAR FORMS.** 1. If  $(X, D; \Gamma)$  is a convex space, then Theorem 2 reduces to Park [P5, Theorem 7] which includes many known generalizations of the KKM theorem.

2. For  $H$ -spaces, Theorem 2 generalizes Horvath [H1, Théorème 3.1 et Corollaire 3], [H3, I, Theorem 1 and Corollary 1], Bardaro and Cepitelli [BC1, Theorem 1], Ding and Tan [DT, Corollary 1 and Theorem 8], Ding, Kim, and Tan [DKT1, Lemma 1], and Park [P1, Theorems 1 and 4], [P2, Theorem 1].

**REMARKS.** 1. If  $(X, D; \Gamma)$  is a convex space with  $\Gamma_A = \text{co } A$  for  $A \in \langle D \rangle$ , then (i) implies (ii). In fact, we can choose  $L_N = \text{co}(M \cup N)$ .

2. Ding [Di, Theorem 3.2] claimed that Theorem 2 for Case (ii) holds for a compact-valued u.s.c. map  $F$  and

$$(2.2)' \text{ for each } N \in \langle D \rangle, \Gamma_N \subset F^{-}G(N)$$

instead of (2.2). However, this is false as the following example shows:

**EXAMPLE.** Let  $X = Y = K = [0, 1]$ ,  $D = \{0, 1\}$ , and  $\Gamma_N = \text{co } N$  for  $N \in \langle D \rangle$ . Let  $Gx = \{x\}$  for  $x \in D$  and  $Fx = [0, 1]$  for  $x \in X$ . Then  $\bigcap \{Gx : x \in D\} = \emptyset$  and for each  $N \in \langle D \rangle$ ,  $\text{co } N \subset F^{-}G(N) = [0, 1]$ .

### 5. Fixed point theorems

Our main coincidence theorem can be applied to some fixed point problems for the class  $\mathfrak{A}_c$ .

**THEOREM 3.** *Let  $(X, D; \Gamma)$  be an  $H$ -space such that  $X$  has a Hausdorff uniform structure and  $F \in \mathfrak{A}_c(X, X)$  a compact map. Suppose that, for each entourage  $V$  of  $X$ , there exist maps  $S : D \rightarrow X$  and  $T : X \rightarrow X$  satisfying (1)-(3) in Corollary 1 and  $\text{Gr}(F) \cap \text{Gr}(T) \subset V$ . Then  $F$  has a fixed point.*

*Proof.* For each entourage  $V$  of the Hausdorff uniformity, by Corollary 1, there exist a map  $T : X \rightarrow X$  and a point  $(x_V, y_V)$  such that



$(x_V, y_V) \in \text{Gr}(F) \cap \text{Gr}(T) \subset V$ . Therefore,  $F$  has a  $V$ -fixed point. Since  $\overline{F(X)}$  is compact and  $F$  is u.s.c.,  $F$  must have a fixed point.

**PARTICULAR FORMS.** See Horvath [H5, Theorem 4.4] and Park [P3, Theorem 5].

For metric spaces, we have the following from Theorem 3:

**THEOREM 4.** Let  $(X, D; \Gamma)$  be a metric  $H$ -space with the metric  $d$ , and  $F \in \mathfrak{A}_c(X, X)$  a compact map. Suppose that

(4.1) every open ball in  $X$  is  $H$ -convex; and

(4.2)  $D$  is dense in  $\overline{F(X)}$ .

Then  $F$  has a fixed point.

*Proof.* Note that  $X$  is a Hausdorff uniform space. For any  $\varepsilon > 0$  and an entourage  $V_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$ , define  $S : D \rightarrow X$  and  $T : X \rightarrow X$  by

$$Sx = B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\} \quad \text{for } x \in D,$$

and

$$Tx = B(x, \varepsilon) \quad \text{for } x \in X.$$

Note that  $Sx = Tx$  for  $x \in D$  and  $Sx$  is open in  $X$ . Moreover, for each  $y \in X$ ,

$$T^{-1}y = \{x \in X : y \in B(x, \varepsilon)\} = B(y, \varepsilon)$$

is  $H$ -convex. Furthermore,  $\overline{F(X)} \subset S(D)$  by (4.2). Note that

$$\text{Gr}(T) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\} = V_\varepsilon.$$

Therefore, by Theorem 3,  $F$  has a fixed point.

**PARTICULAR FORMS.** 1. For  $\mathfrak{A} = \mathbb{C}$ , Theorem 4 extends Brouwer [Br], Schauder [Sc], Rassias [R], and Park [P3, Theorems 3 and 6]. For details, see [P3].

2. For  $\mathfrak{A} = \mathbb{K}$ , Theorem 4 extends Kakutani [Ka] and Bohnenblust and Karlin [BK].

### 6. Open-valued KKM theorems and coincidence theorems

In order to obtain the open version of Theorem 2 for the class  $\mathcal{A}_c$ , we need the following:

**THEOREM 5.** *Let  $(X, D; \Gamma)$  be an  $H$ -space,  $D = \{x_0, x_1, \dots, x_n\} \in \langle X \rangle$ ,  $Y$  a regular space,  $G : D \rightarrow Y$  a map, and  $F : X \rightarrow Y$  a compact valued u.s.c. map. If  $G : D \rightarrow Y$  is an open valued map such that*

$$(5.1) \text{ for each } J \in \langle D \rangle, F(\Gamma_J) \subset G(J).$$

*Then there is a closed valued map  $H : D \rightarrow Y$  such that  $Hx \subset Gx$  for all  $x \in D$  and*

$$(5.2) \text{ } Ff(\Delta_J) \subset H(J) \text{ for each } J \in \langle D \rangle \text{ and } f : \Delta_n \rightarrow \Gamma_D \text{ in Lemma, where } \Delta_J \text{ is the face of } \Delta_n \text{ corresponding to } J.$$

*Proof.* For any  $y \in G(D)$ , let

$$H_y = \bigcap \{Gx : y \in Gx\};$$

then  $H_y$  is an open set in  $Y$  containing  $y$ . By the regularity of  $Y$ , there exists an open neighborhood  $U_y$  of  $y$  in  $Y$  such that

$$y \in U_y \subset \overline{U_y} \subset H_y.$$

Clearly for any  $J \in \langle D \rangle$ , we have

$$G(J) = \{U_y : y \in G(J)\}$$

and so by (5.1),  $\bigcup \{U_y : y \in G(J)\}$  is an open cover of  $Ff(\Delta_J)$ , since  $Ff(\Delta_J) \subset F(\Gamma_J)$ .

Since  $Ff(\Delta_J)$  is compact, there exists a  $B_J \in \langle G(J) \rangle$  such that

$$Ff(\Delta_J) \subset \bigcup \{U_y : y \in B_J\}.$$

Let  $B = \bigcup \{B_J : J \in \langle D \rangle\}$ . Define  $H : D \rightarrow Y$  by

$$Hx = \bigcup \{\overline{U_y} : y \in B \cap Gx\}$$

for each  $x \in D$ . Then  $Hx$  is closed in  $Y$  for each  $x \in D$  and  $Hx \subset Gx$ , since  $\overline{U_y} \subset H_y \subset Gx$  if  $y \in Gx$ . And for each  $J \in \langle D \rangle$  and  $z \in Ff(\Delta_J)$ , we have  $z \in U_y$  for some  $y \in B_J \subset G(J) \cap B$ ; that is  $y \in Gx \cap B$  for some  $x \in J$ , hence

$$Ff(\Delta_J) \subset H(J).$$

This completes our proof.

REMARK. The proof was motivated by Shih [Sh, Theorem 1].

THEOREM 6. Let  $(X, D; \Gamma)$  be an  $H$ -space,  $D = \{x_0, x_1, \dots, x_n\} \in \langle X \rangle$ ,  $Y$  a  $T_1$  regular space,  $G : D \dashrightarrow Y$  a map, and  $F \in \mathfrak{A}_c(X, Y)$ . Suppose that

(6.1) for each  $x \in D$ ,  $Gx$  is open in  $Y$ ; and

(6.2) for each  $N \in \langle D \rangle$ ,  $F(\Gamma_N) \subset G(N)$ .

Then  $F(\Gamma_D) \cap \bigcap \{Gx : x \in D\} \neq \emptyset$ .

*Proof.* Suppose that  $F(\Gamma_D) \cap \bigcap \{Gx : x \in D\} = \emptyset$ . By Theorem 5, there exists a map  $H : D \dashrightarrow Y$  such that  $Hx \subset Gx$  for each  $x \in D$  and (5.2) holds, and so  $F(\Gamma_D) \cap \bigcap \{Hx : x \in D\} = \emptyset$ . Then  $Ff(\Delta_n) \subset T(D) = Y$ , where  $Tx = Y \setminus Hx$  for each  $x \in D$ . Let  $\{\lambda_i\}_{i=0}^n$  be the partition of unity subordinate to this cover  $\{Tx_i\}_{i=0}^n$  of compact subset  $Ff(\Delta_n)$  of  $Y$ . Define  $p : Ff(\Delta_n) \rightarrow \Delta_n$  by

$$py = \sum_{i=0}^n \lambda_i(y)e_i = \sum_{i \in N_y} \lambda_i(y)e_i$$

for  $y \in Ff(\Delta_n)$  where  $i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in Tx_i$ .

By Lemma,  $pFf \in \mathfrak{A}_c(\Delta_n, \Delta_n)$  has a fixed point  $z_0 \in \Delta_n$ ; that is  $z_0 \in pFfz_0$ . So there is a  $y_0 \in Ffz_0$  such that  $py_0 = z_0$  and  $y_0 \in Fx_0$  where  $x_0 = fz_0$ . If  $i \in N_{y_0}$ , then  $y_0 \in Tx_i$ , and

$$y_0 \in Fx_0 \cap \bigcap \{Tx_i : i \in N_{y_0}\} \neq \emptyset.$$

Put  $M = \{x_i : i \in N_{y_0}\}$ , then  $Ff(\Delta_M) \cap \bigcap \{Tx : x \in M\} \neq \emptyset$ ; that is,  $Ff(\Delta_M) \not\subset H(M)$ , where  $\Delta_M$  is the face of  $\Delta_n$  corresponding to  $M$ . This contradicts (5.2). This completes our proof.

REMARKS. 1. In the proof of Theorem 6, we actually showed the following:

Let  $X$  be a topological space,  $f : \Delta_n \rightarrow X$  a continuous function, and  $Y, D, F$  and  $G$  be the same as in Theorem 6. Suppose that (6.1) and

(6.2)' for each  $N \subset D$ ,  $Ff(\Delta_N) \subset G(N)$ .

Then  $Ff(\Delta_n) \cap \bigcap \{Gx : x \in D\} \neq \emptyset$ .

2. If  $F$  is a continuous single-valued map, then the  $T_1$  regularity of  $Y$  is not necessary. See Park [P1, Theorem 14].

PARTICULAR FORMS. For convex spaces, Theorem 6 extends W.K. Kim [Kil-2], Lassonde [L2, Theorem 1], and Park [P4, Theorem 10].

By the same argument to that of Theorems 1 and 2, we can show that Theorem 6 is equivalent to the following:

THEOREM 7. Let  $(X, D; \Gamma)$  be an  $H$ -space,  $D \in \langle X \rangle$ ,  $Y$  a  $T_1$  regular space, and  $F \in \mathfrak{A}_c(X, Y)$ . Let  $S : D \rightarrow Y$  and  $T : X \rightarrow Y$  satisfy the following:

- (7.1) for each  $x \in D$ ,  $Sx$  is closed in  $Y$ ;
- (7.2) for each  $y \in F(X)$ ,  $M \in \langle S^{-1}y \rangle$  implies  $\Gamma_M \subset T^{-1}y$ ; and
- (7.3)  $F(\Gamma_D) \subset S(D)$ .

Then  $T$  and  $F$  have a coincidence point  $\bar{x} \in X$ ; that is,  $T\bar{x} \cap F\bar{x} \neq \emptyset$ .

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