Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 24, No. 11, pp. 1549 1554, 1995

0362-546X/95 $9.50 + .00

0362-546X(94)00095-6

ACYCLIC MAPS, MINIMAX INEQUALITIES AND FIXED POINTS

SEHIE PARK†

Department of Mathematics, Seoul National University and The Mathematical Sciences Research Institute of Korea, Seoul 151-742, Korea

(*Received* 1 *December* 1991; *received for publication* 5 *May* 1994)

Key words and phrases: Convex space, acyclic space, upper semicontinuous (u.s.c.) multifunction, acyclic map, Hausdorff locally convex space, quasi-concave [-convex], inward [outward] sets, sequentially u.s.c. with respect to weak topology.

A *convex space* is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

For topological spaces X and Y, a multifunction $T: X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each open subset G of Y, the set $\{x \in X \mid Tx \subset G\}$ is open in X. Here, 2^Y denotes the class of nonempty subsets of Y. We introduce two classes of multifunctions $T: X \to 2^Y$ as follows [1]

$T \in \mathbf{K}(X, Y) \Leftrightarrow$ T is an u.s.c. multifunction with nonempty compact convex values, where Y is a convex space.

$T \in \mathbf{V}(X, Y) \Leftrightarrow$ T is an acyclic map, that is, an u.s.c. multifunction with compact acyclic values.

As usual, the set $\{(x, y) \mid y \in Tx\}$ is called either the graph of T, or, simply, T.

In this paper, we obtain fixed point theorems for acyclic maps in $\mathbf{V}(X, E)$ generalizing corresponding ones for maps in $\mathbf{K}(X, E)$ with certain boundary conditions, where X is a compact convex subset of a Hausdorff locally convex topological vector space E. Consequently, we generalize results in [2-9], and many others. We mainly follow the method of Ha [5] and Park [7, 10].

Recall that an extended real-valued function $f: X \to \bar{\mathbf{R}}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* if $\{x \in X \mid fx > r\}$ [resp. $\{x \in X \mid fx < r\}$] is open for each $r \in \bar{\mathbf{R}}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp. *quasiconvex*] whenever $\{x \in X \mid fx > r\}$ [resp. $\{x \in X \mid fx < r\}$] is convex for each $r \in \bar{\mathbf{R}}$.

We begin with the following minimax inequality with respect to an acyclic map. This is actually a consequence of the Lefschetz fixed point theory. For the literature, see [1].

THEOREM 1. Let X be a convex space, Y a Hausdorff space, $T \in \mathbf{V}(X, Y)$ a compact multifunction, and $\phi, \psi: X \times Y \to \bar{\mathbf{R}}$ two extended real-valued functions such that:
(1) $\phi(x, y) \le \psi(x, y)$ for each $(x, y) \in X \times Y$;
(2) for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semicontinuous on Y; and
(3) for each $y \in Y$, $x \mapsto \psi(x, y)$ is quasiconcave on X.

† Supported in part by the Basic Sciences Research Institute Program, Ministry of Education, 1991.

Then there exists a $\bar{y} \in Y$ such that

$$\sup_{x \in X} \phi(x, \bar{y}) \leq \sup_{(x,y) \in T} \psi(x, y).$$

Remark. Theorem 1 is essentially due to Granas and Liu [1, theorem 7.1] as a generalization of the celebrated 1972 minimax inequality of Ky Fan [11]. When $\phi = \psi$, $T \in \mathbf{K}(X, Y)$, and Y is a compact convex space, theorem 1 reduces to Ha [5, theorem 1], where the Hausdorffness of X and the convexity of Y are superfluous. A far-reaching generalization of theorem 1 is obtained in a recent work of the author [12].

The following is a variant of theorem 1 with a lopsided saddle point.

THEOREM 2. Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathbf{V}(X, Y)$. Let $g: X \times Y \to \mathbf{R}$ be a continuous function such that for each $y \in Y$, $x \mapsto g(x, y)$ is quasiconvex on X. Then there exists an $(x_0, y_0) \in T$ such that

$$g(x_0, y_0) \leq g(x, y_0) \qquad \text{for all } x \in X.$$

Proof. Define $\phi: X \times Y \to \mathbf{R}$ by

$$\phi(x, y) = \min_{z \in X} g(z, y) - g(x, y)$$

for $(x, y) \in X \times Y$. Then it is easy to see that ϕ is continuous on $X \times Y$ [13, p. 70] and satisfies (1), (2) and (3) of theorem 1 with $\phi = \psi$. Moreover, T is compact since it is u.s.c. and compact-valued. Therefore, by theorem 1, there exists a $\bar{y} \in Y$ such that

$$\sup_{x \in X} \phi(x, \bar{y}) \leq \sup_{(x,y) \in T} \phi(x, y).$$

Since $x \mapsto g(x, \bar{y})$ is continuous on the compact set X, there exists an $\bar{x} \in X$ such that $g(\bar{x}, \bar{y}) = \min_{z \in X} g(z, \bar{y})$ or $\phi(\bar{x}, \bar{y}) = 0$. Hence, we have

$$0 \leq \sup_{(x,y) \in T} \phi(x, y).$$

Since the graph of T is closed and hence compact in $X \times Y$, the supremum in the above inequality is attained. This completes our proof.

Remark. For $T \in \mathbf{K}(X, Y)$, theorem 2 reduces to Ha [5, theorem 2], where the Hausdorffness of X and the convexity of Y are superfluous. For $X = Y$ and $T = 1_X$, the identity function of X, theorem 2 reduces to Fan [11, corollary 1]. Moreover, for a normed vector space $E = Y$, a single-valued T, and $g(x, y) = \|x - y\|$, theorem 2 reduces to Fan [14, theorem 2].

For a subset X of a topological vector space E and $x \in X$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows [15]

$$I_X(x) := \{x + r(u - x) \in E \mid u \in X,\ r > 0\},$$

$$O_X(x) := \{x - r(u - x) \in E \mid u \in X,\ r > 0\}.$$

The closures of $I_X(x)$ and $O_X(x)$ are denoted by $\bar{I}_X(x)$ and $\bar{O}_X(x)$, resp. In the sequel, $W(x)$ denotes either $\bar{I}_X(x)$ or $\bar{O}_X(x)$.

Let $\mathcal{P}$ denote the family of all continuous seminorms on a Hausdorff locally convex space E.

As an application of theorem 2, we obtain the following Ky Fan type fixed point theorem for acyclic maps.

THEOREM 3. Let X be a compact convex subset of a Hausdorff locally convex space E and $T \in \mathbf{V}(X, E)$. Then either T has a fixed point or there exist an $(x_0, y_0) \in T$ and a $p \in \mathcal{P}$ such that

$$0 < p(x_0 - y_0) \leq p(x - y_0) \qquad \text{for all } x \in W(x_0).$$

Proof. Suppose T has no fixed point. Then for each $x \in X$, the origin 0 of E does not belong to the compact set $x - Tx$ and so there exists a $\delta_x > 0$ and a $p_x \in \mathcal{P}$ such that $p_x(x - y) > 2\delta_x$ for all $y \in Tx$. Since T is u.s.c., there exists an open neighbourhood U_x of x in X such that $p_x(z - v) > \delta_x$ for all $z \in U_x$ and $v \in Tz$. Since $\{U_x \mid x \in X\}$ covers X, there exists a finite subset N of X such that $\{U_x \mid x \in N\}$ covers X. Let $p = \max\{p_x \mid x \in N\}$ and $\delta = \min\{\delta_x \mid x \in N\} > 0$. Then $p \in \mathcal{P}$ and $p(x - y) > \delta$ for all $(x, y) \in T$.

We define a function $g: X \times Y \to \mathbf{R}$ by $g(x, y) = p(x - y)$ for $(x, y) \in X \times Y$, where $Y := T(X)$ which is compact. Then clearly g and T satisfy all of the requirements of theorem 2. Therefore, there exists an $(x_0, y_0) \in T$ such that

$$0 < p(x_0 - y_0) \leq p(x - y_0) \qquad \text{for all } x \in X.$$

For $x \in I_X(x_0) \backslash X$, there exists $u \in X$ and $r > 1$ such that $x = x_0 + r(u - x_0)$. Suppose that $p(x - y_0) < p(x_0 - y_0)$. Since

$$\frac{1}{r}x + \left(1 - \frac{1}{r}\right)x_0 = u \in X,$$

we have

$$p(x - y_0) \leq \frac{1}{r}p(x - y_0) + \left(1 - \frac{1}{r}\right)p(x_0 - y_0) < p(x_0 - y_0),$$

which is a contradiction. Therefore, $p(x_0 - y_0) \leq p(x - y_0)$ holds for all $x \in I_X(x_0)$, and hence, for all $x \in \bar{I}_X(x_0)$.

For the outward case, consider the map $T' \in \mathbf{V}(X, E)$ given by $T'x = 2x - Tx$ for each $x \in X$. Then, by the above inward case, there exists an $(x_0, y_1) \in T'$ and a $p \in \mathcal{P}$ such that

$$0 < p(x_0 - y_1) \leq p(x' - y_1) \qquad \text{for all } x' \in I_X(x_0).$$

For $x \in O_X(x_0)$, let $x' = 2x_0 - x$ and $y_1 = 2x_0 - y_0$ where $y_0 \in Tx_0$. Then we have

$$0 < p(x_0 - y_0) \leq p(x - y_0) \qquad \text{for all } x \in O_X(x_0),$$

and hence, for all $x \in \bar{O}_X(x_0)$. This proves theorem 3 for the case $W(x_0) = \bar{O}_X(x_0)$.

Remarks. (1) For $T \in \mathbf{K}(X, E)$, theorem 3 reduces to Park [7, theorem 2] and Reich [8, theorem 2], and improves Ha [5, theorem 3] and Fan [14, theorem 1].

(2) Note that the x_0 in the conclusion of theorem 3 belongs to the boundary of X, $\operatorname{Bd} X$. In fact, suppose that $x_0 \in \operatorname{Int} X$. Then x_0 is an internal point and $W(x_0) = E$. By putting $x = y_0$, we have $0 < p(x_0 - y_0) \leq 0$ in the conclusion of theorem 3, which is a contradiction.

As a direct consequence of theorem 3, we have the following theorem.

THEOREM 4. Let X be a compact convex subset of a Hausdorff locally convex space E and $T \in \mathbf{V}(X, E)$. If T satisfies one of the following conditions, then T has a fixed point.

For each $x \in \operatorname{Bd} X \backslash Tx$:

(0) for each $y \in Tx$ and each $p \in \mathcal{P}$, $p(y - x) > 0$ implies $p(y - x) > p(y - z)$ for some $z \in W(x)$;

(i) for each $y \in Tx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that

$$|\lambda| < 1 \qquad \text{and} \qquad \lambda x + (1 - \lambda)y \in W(x);$$

(ii) $Tx \subset W(x)$;

(iii) for each $y \in Tx$, ther exists a number λ (as in (i)) such that

$$|\lambda| < 1 \qquad \text{and} \qquad \lambda x + (1 - \lambda)y \in X;$$

(iv) $Tx \subset IF_X(x) = \{x + c(u - x) \mid u \in X, \operatorname{Re}(c) > 1/2\}$;

(iv)′ $Tx \subset OF_X(x) = \{x + c(u - x) \mid u \in X, \operatorname{Re}(c) < -1/2\}$;

(v) $Tx \subset X$;

(vi) $T(X) \subset X$.

Proof. (0) Clear from theorem 3 and remark 2 above.

(i) For any $p \in \mathcal{P}$ satisfying $p(y - x) > 0$, put $z = \lambda x + (1 - \lambda)y$ in (0). Then we have

$$p(y - z) = p(\lambda y - \lambda x) = |\lambda| p(y - x) < p(y - x)$$

since $|\lambda| < 1$.

(ii) If $Tx \subset W(x)$, then for each $y \in Tx$, we can choose $\lambda = 0$ in (i).

(iii) Since $X \subset I_X(x)$, we clearly have (iii) $\Rightarrow$ (i).

(iv) Note that (iv) $\Leftrightarrow$ (iii) [16].

(iv)′ This is a dual form of (iv) [16].

(v) If $Tx \subset X$, then for each $y \in Tx$, we can choose $\lambda = 0$ in (iii).

(vi) Clearly, we have (vi) $\Rightarrow$ (v).

Remarks. (1) To the best of our knowledge, only case (vi) of theorem 4 is known. In fact, since X is an *lc* space, theorem 4 (vi) follows from Begle [17, theorem 1]. Note that "each $y \in Tx$" cannot be replaced by "there is $y \in Tx$", as noted by Reich [16, example 1.2]. For some other conditions equivalent to (0), see [18].

(2) Following Halpern [24], for a subset D of a normed vector space E, we define

$$n_D(x) = \{y \in E \mid y \neq x, \|y - x\| \leq \|y - z\| \text{ for all } z \in D\}$$

for $x \in D$, and consider a "nowhere normal outward" multifunction $T: D \to 2^X$, that is,

(0)′ $$Tx \cap n_D(x) = \varnothing \qquad \text{for } x \in D.$$

Then (0)′ clearly implies (0). Particular forms of theorem 4 with condition (0)′ for a normed vector space are given by Fitzpatrick and Petryshyn [3, theorem 3 (i)], Reich [8, theorem 3.3 (a)], Halpern [19, theorem 20], and Halpern and Bergman [15, theorem 2.1].

(3) Particular forms of theorem 4 (ii) are given by Fitzpatrick and Petryshyn [3, corollary 1] and Halpern [19, corollaries 21 and 22]. Moreover, for the outward case in theorem 4 (ii), we have the surjectivity $X \subset T(X)$ as in [12, 20], and Halpern [19, corollary 23]. Note that Halpern [19, theorem 19] is a simple consequence of theorem 4 (vi).

(4) For a $T \in \mathbf{K}(X, E)$, if E is real or if T is single-valued, then theorem 4 has far-reaching generalizations. See Park [12, 21, 22].

(5) However, for a $T \in \mathbf{K}(X, E)$, single-valued or multi-valued, each case of theorem 4 generalizes historically well-known results as follows:

(0) Reich [18, theorem 7; 9, theorems 1 and 2] and Browder [23, corollary to theorem 9];

(i) Park [7, theorem 4];

(ii) Browder [24, theorems 1 and 2];

(iii) Fan [14, theorem 3] and Ha [5, theorem 4];

(iv), (iv)′ Reich [8, theorem 3.1];

(v) Rothe [25];

(vi) Brouwer [26], Schauder [17, Satz 1], Tychonoff [28, Satz], Kakutani [6, theorem 1], Bohnenblust and Karlin [29], Glicksberg [4, theorem], and Fan [2, theorem 1].

(6) As in Reich [9], condition (0) can be reformulated using the subdifferential ∂p of p. Moreover, as Reich [9] noted, theorem 4 is also valid for lower semicontinuous $T: X \to 2^E$ if E is completely metrizable and if T has closed convex values. This follows from the Michael selection theorem and theorem 4.

(7) In theorem 4, since T is compact, if E is real, then upper semicontinuity of T can be replaced by upper demicontinuity or upper hemicontinuity. For the literature, see [20, 21].

Finally, we have the following corollary.

Corollary. Let X be a weakly compact convex subset of a metrizable locally convex space E and $T: X \to 2^E$ a multifunction with weakly compact acyclic values. If T is sequentially u.s.c. with respect to the weak topology and satisfies one of (0)–(vi) in theorem 4, then T has a fixed point.

Proof. It suffices to show that T is weakly u.s.c., so that theorem 4 works. In fact, for each weakly closed subset C of E, $T^{-1}(C)$ is sequentially closed for the weak topology on X by assumption, hence weakly compact by the Eberlein–Šmulian theorem, and $T^{-1}(C)$ is weakly closed. Therefore, T is weakly u.s.c.

Remark. When T is convex-valued, corollary for case (vi) reduces to the multi-valued version of Arino *et al.* [30, theorem 1]. When T is single-valued, corollary for case (ii) is due to Park [22, corollary 2].

REFERENCES

1. GRANAS A. & LIU F.-C., Coincidences for set-valued maps and minimax inequalities, *J. Math. pures appl.* **65**, 119–148 (1986).
2. FAN K., Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. natn. Acad. Sci. U.S.A.* **38**, 121–126 (1952).
3. FITZPATRICK P. M. & PETRYSHYN W. V., Fixed point theorems for multivalued acyclic mapping, *Pacif. J. Math.* **54**, 17–23 (1974).

4. GLICKSBERG I. L., A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, *Proc. Am. math. Soc.* **3**, 170–174 (1952).
5. HA C.-W., On a minimax inequality of Ky Fan, *Proc. Am. math. Soc.* **99**, 680–682 (1987).
6. KAKUTANI S., A generalization of Brouwer's fixed point theorem, *Duke math. J.* **8**, 457–459 (1941).
7. PARK S., Fixed point theorems on compact convex sets in topological vector spaces, II, *J. Korean math. Soc.* **26**, 175–179 (1989).
8. REICH S., Approximate selections, best approximations, fixed points, and invariant sets, *J. math. Analysis Applic.* **62**, 104–113 (1978).
9. REICH S., Fixed point theorems for set-valued maps, *J. math. Analysis Applic.* **69**, 353–358 (1979).
10. PARK S., Fixed point theorems on compact convex sets in topological vector spaces, *Contemp. Math.* **72**, 183–191 (1988).
11. FAN K., A minimax inequality and applications, in *Inequalities III* (Edited by O. SHISHA), pp. 103–113. Academic Press, New York (1972).
12. PARK S., Some coincidence theorems on acyclic multifunctions and applications to KKM theory, *Proc. of 2nd Int. Conf. on Fixed Point Theory and Applications, Halifax, 9–14 June* 1991 (Edited by K.-K. TAN), pp. 248–277. World Sci. Publ., River Edge, NJ (1993).
13. AUBIN J.-P., *Mathematical Methods of Game and Economic Theory,* Rev. Ed. North-Holland, Amsterdam (1982).
14. FAN K., Extensions of two fixed point theorems of F. E. Browder, *Math. Z.* **112**, 234–240 (1969).
15. HALPERN B. & BERGMAN G. E., A fixed-point theorem for inward and outward maps, *Trans. Am. math. Soc.* **13**, 353–358 (1968).
16. REICH S., Fixed points in locally convex spaces, *Math. Z.* **125**, 17–31 (1972).
17. BEGLE E. G., A fixed point theorem, *Ann. Math.* **51**, 544–550 (1950).
18. REICH S., Some problems and results in fixed point theory, *Contemp. Math.* **21**, 179–187 (1983).
19. HALPERN B., Fixed-point theorems for set-valued maps in infinite dimensional spaces, *Math. Ann.* **189**, 87–98 (1970).
20. PARK S., Fixed point theory of multifunctions in topological vector spaces, *J. Korean math. Soc.* **29**, 191–208 (1992).
21. PARK S., Fixed point and coincidence theorems for u.h.c. multifunctions on convex sets, *Proc. Coll. Natur. Sci. SNU* **15**, 1–15 (1990).
22. PARK S., A generalization of the Brouwer fixed point theorem, *Bull. Korean math. Soc.* **28**, 43–47 (1991).
23. BROWDER F. E., Coincidence theorems, minimax theorems, and variational inequalities, *Contemp. Math.* **26**, 67–80 (1984).
24. BROWDER F. E., A new generalization of the Schauder fixed point theorem, *Math. Ann.* **174**, 285–290 (1967).
25. ROTHE E. H., Zur Theorie der topologischen Ordnung und der Vektorfelder in Banachschen Räumen, *Comp. Math.* **5**, 177–197 (1937)
26. BROUWER L. E. J., Über Abbildungen von Mannigfaltigkeiten, *Math. Annln* **71**, 97–115 (1912).
27. SCHAUDER J., Der Fixpunktsatz in Funktionalräumen, *Studia math.* **2**, 171–180 (1930).
28. TYCHONOFF A., Ein Fixpunktsatz, *Math. Annln* **111**, 767–776 (1935).
29. BOHNENBLUST H. F. & KARLIN S., On a theorem of Ville, in *Contributions to the Theory of Games, Ann. Math. Studies*, No. 24, pp. 155–160. Princeton University Press, NJ (1950).
30. ARINO O., GAUTIER S. & PENOT J. P., A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations, *Funkcialaj Ekvacioj* **27**, 273–279 (1984).