FIXED POINTS OF CONDENSING MAPS ON SPHERES SATISFYING THE LERAY-SCHAUDER CONDITION

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Abstract. We show that a continuous condensing non-selfmap on a sphere in an infinite dimensional Banach space has a fixed point if it satisfies the Leray-Schauder condition or other boundary condition.

In this paper, we strengthen and extend results of Lin [3] on approximations and fixed points for condensing non-selfmaps defined on a sphere in a Banach space.

A continuous selfmap defined on a sphere in a Banach space may not have fixed points. Nussbaum [7] proved that a continuous k-set-contraction from a sphere into itself has a fixed point if the space is infinite dimensional. Massatt [5] generalized this result to continuous condensing maps. Recently, Lin [3] extended this result to non-selfmaps on spheres using the following best approximation theorem:

Theorem 1. [3] *Let S_r be a sphere with center at origin and radius $r > 0$ in an infinite-dimensional Banach space E. Let $f : S_r \to E$ be a continuous condensing map satisfying*

$$\|fx\| \geq r \quad \text{for each } x \in S_r. \tag{0}$$

Then there exists a point $u \in S_r$ such that

$$\|u - fu\| = d(fu, S_r) = d(fu, B_r).$$

Here, $B_r = \{x \in E : \|x\| \leq r\}$, $S_r = \operatorname{Bdry} B_r$, and $d(x, A) = \inf_{y \in A} \|x - y\|$ for $A \subset E$. For definitions of condensing maps and k-set-contractions with respect to the Kuratowski measure of noncompactness, see [4, 6]. For a subset $X \subset E$ and $x \in X$, the inward set of X at x is defined as

$$I_X(x) = \{x + c(z - x) : z \in X,\ c > 0\},$$

and its closure $\overline{I}_X(x)$ is called the weakly inward set [1].

We strengthen Theorem 1 as follows:

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Theorem 2. *Let S_r, E, and f be the same as in Theorem 1. Then either f has a fixed point $u \in S_r$ or there exists a point $u \in S_r$ such that*

$$0 < \|u - fu\| = d(fu, \overline{I}_{B_r}(u)).$$

Proof. By Theorem 1, we have a $u \in S_r$ such that $\|u - fu\| = d(fu, B_r)$. If $\|u - fu\| = 0$, then u is a fixed point. Suppose $0 < \|u - fu\|$. We show that

$$\|u - fu\| \leq \|fu - z\| \quad \text{for all } z \in I_{B_r}(u).$$

In fact, for $z \in I_{B_r}(u) \backslash B_r$, there exist $x \in B_r$ and $c > 1$ such that $z = u + c(x - u)$. Suppose that $\|u - fu\| > \|fu - z\|$. Since

$$\frac{1}{c} z + (1 - \frac{1}{c}) u = x \in B_r,$$

we have

$$\|fu - x\| \leq \frac{1}{c}\|fu - z\| + (1 - \frac{1}{c})\|fu - u\| < \|u - fu\|,$$

which contradicts $\|u - fu\| = d(fu, B_r)$. Moreover, since $\| \ \|$ is continuous, we have

$$\|u - fu\| \leq \|fu - z\| \quad \text{for all } z \in \overline{I}_{B_r}(u).$$

This completes our proof.

From Theorem 2, we have the following fixed point theorem:

Theorem 3. *Let S_r, E, and f be the same as in Theorem 1. Then f has a fixed point whenever one of the following conditions is satisfied for $x \in S_r$ such that $x \neq fx$:*

(i) *There exists a $y \in \overline{I}_{B_r}(x)$ satisfying*

$$\|y - fx\| < \|x - fx\|.$$

(ii) *There exists a number λ (real or complex, depending on whether X is real or complex) such that $|\lambda| < 1$ and*

$$\lambda x + (1 - \lambda) fx \in \overline{I}_{B_r}(x).$$

(iii) $fx \in \overline{I}_{B_r}(x)$

(iv) *There exists a* $y \in B_r$ *satisfying*

$$\|y - fx\| < \|x - fx\|.$$

(v) $\lim_{h \to 0^+} d[(1-h)x + h(fx),\ B_r]/h = 0.$

(vi) *There exists a number* λ *(as in* (ii)*) such that*

$$\lambda x + (1-\lambda) fx \in B_r.$$

(vii) $fx \in IF_{B_r}(x) = \{x + c(y-x) \in E : y \in B_r,\ \mathrm{Re}(c) > \frac{1}{2}\}.$

(viii) $f(S_r) \subset S_r$.

Proof. (i) Suppose that f has no fixed point. Then, by Theorem 2, there exists a $u \in S_r$ satisfying

$$0 < \|u - fu\| = d(fu, \overline{I}_{B_r}(u)).$$

On the other hand, there exists a $y \in \overline{I}_{B_r}(u)$ satisfying

$$\|y - fu\| < \|u - fu\|.$$

This is a contradiction.

(ii) Let $y = \lambda x + (1-\lambda) fx$. If $x \neq fx$, then

$$\|y - fx\| = \|\lambda x - \lambda fx\| = |\lambda|\, \|x - fx\| < \|x - fx\|$$

since $|\lambda| < 1$. Therefore, (ii) $\Longrightarrow$ (i)

(iii) For $\lambda = 0$, $\lambda x + (1-\lambda) fx = fx \in \overline{I}_{B_r}(x)$. Hence, (iii) $\Longrightarrow$ (ii).

(iv) Since $y \in B_r \subset \overline{I}_{B_r}(x)$, (iv) $\Longrightarrow$ (i).

(v) It is well-known that (iii) $\Longleftrightarrow$ (v). See [8] for references.

(vi) It is clear that (vi) $\Longrightarrow$ (ii).

(vii) It is well-known that (vii) $\Longleftrightarrow$ (vi). See [8].

(viii) Note that (viii) implies any of (i)-(vii).

Let $p : E \backslash (\mathrm{Int}\, B_r) \to S_r$ be the radial projection; that is, $p(x) = rx/\|x\|$ for $x \in E \backslash (\mathrm{Int}\, B_r)$. Then, by Nussbaum [6, Corollary A.1], p is a continuous 1-set-contraction.

We have another fixed point theorem for maps satisfying the so-called Leray-Schauder boundary condition:

Theorem 4. *Let S_r, E, and f be the same as in Theorem 1. Then f has a fixed point whenever one of the following conditions is satisfied:*

(ix) $fx \neq \alpha x$ *for each* $x \in S_r$ *and* $\alpha > 1$.

(x) $\|fx - x\|^2 \geq \|fx\|^2 - r^2$ *for each* $x \in S_r$.

(xi) $\|x - fx\| > \|fx\|$ *for each* $x \in S_r$, $x \neq fx$.

Proof. (ix) Let $g = p \circ f : S_r \to S_r$. Then g is a continuous condensing map. From Theorem 1 replacing f by g, we have a point $u \in S_r$ such that

$$\|u - gu\| = d(gu, S_r) = 0.$$

Therefore,

$$u = gu = \frac{r}{\|fu\|} fu \text{ and } \frac{1}{r}\|fu\|u = fu.$$

Hence, by (ix), $\frac{1}{r}\|fu\| \leq 1$ or $\|fu\| \leq r$. On the other hand, $\|fu\| \geq r$ by (0). Therefore, $\|fu\| = r$ and hence $u = gu = fu$.

(x) If $fx = \alpha x$ in $\|fx - x\|^2 \geq \|fx\|^2 - r^2$, then $\alpha \leq 1$. Therefore, (x) implies (ix).

(xi) Clearly (xi) $\Longrightarrow$ (x) and (xi) $\Longrightarrow$ (iv) with $y = 0$.

Remarks. 1. Theorem 3(i),(iii) and Theorem 4(x) are due to Lin [3, Theorem 2], and Theorem 3(viii) to Massatt [5].

2. Condition (ix) is the so-called Leray-Schauder boundary condition, (x) is known as the Altman condition, and (iii) is introduced by Halpern [1].

3. The following simple example shows that Theorem 2 does not hold for $\mathbb{R}^n$: Let $f : S_r \to \mathbb{R}^n$ be given By $fx = -2x$ for $x \in S_r$. Then f is continuous and compact (hence, condensing) such that $\|fx\| \geq r$ and $fx \neq \alpha x$ for each $x \in S_r$ and $\alpha > 1$. Clearly, f has no fixed point. Therefore, Theorem 2 characterizes the infinite dimensionality of a Banach space.

Finally, we have another conditions which ensure existence of fixed points for a condensing map defined on a sphere in a Hilbert space:

Theorem 5. *Let S_r be a sphere with center at origin and radius $r > 0$ in an infinite-dimensional Hilbert space E. Let $f : S_r \to E$ be a continuous condensing map satisfying*

$$\|fx\| \geq r \quad \text{for each } x \in S_r. \tag{0}$$

Then f has a fixed point whenever one of the following conditions is satisfied for any $x \in S_r$ with $x \neq fx$:

(xii) *There exists a $z \in B$ such that*

$$\mathrm{Re}\,\langle z - x, fx - x\rangle > 0.$$

(xiii) $\liminf_{h \to 0^+} d[(1-h)x + hfx, B_r]/h < \|x - fx\|$.

(xiv) *There exists a $y \in B_r$ satisfying*

$$\|fx - y\| \leq \|x - y\|.$$

(xv) $\mathrm{Re}\,\langle fx, x\rangle \leq \|x\|^2$.

Proof. For a ball in a Hilbert space, Williamson [9] showed that (xii) $\Longleftrightarrow$ (xiii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (ix). It was shown that (xiv) $\Longrightarrow$ (iv) in [8]. For (xv), if $fx = \alpha x$ in $\mathrm{Re}\,\langle fx, x\rangle \leq \|x\|^2$, then we have $\alpha \leq 1$. Therefore, (xv) $\Longrightarrow$ (ix). This completes our proof.

Remark. For the origin of each of conditions (i)-(xv), see [8]. Those boundary conditions can also be used in Lin [2, Theorems 2,4, and 6].

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