REMARKS ON GENERALIZATIONS OF BEST APPROXIMATION THEOREMS

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INTRODUCTION

The well known best approximation theorems due to Ky Fan [F1] have been of great importance in nonlinear analysis, approximation theory, minimax theory, game theory, and fixed point theory. There have appeared many extensions, variations, and applicaitons of the theorems. See the references in the end of this paper.

One of the most interesting extensions was given by Prolla [Pr] for two functions. Subsequently, a number of its generalizations or variations followed, and some applications to fixed point theory and approximation theory were also given by several authors.

On the other hand, recently there have appeared some best approximation or fixed point theorems for maps whose domains and ranges have different topologies; for example, see [DT, H, Ka, Ki, L, P5, RS, SS1-4, SSS, SSW].

Motivated by those new results, in this paper, we generalize, improve, and unify known Fan or Prolla type best approximation theorems for single-valued maps. Several applications to existence of fixed or coincidence points are also added.

Set-valued generalizations of best approximation theorems will appear in the author's forthcoming work.

PRELIMINARIES

Let $E = (E, \tau)$ be a topological vector space, E^* its topological dual, and $S(E) = S(E, \tau)$ the family of all continuous seminorms on (E, τ). Let w denote the weak topology of E. We say that E^* *separates points of* E if for each $x \in E$ with $x \neq 0$, there exists a $\phi \in E^*$ such that $\phi(x) \neq 0$; that is, if $x \neq 0$ then $p(x) > 0$ for some $p \in S(E, w)$, by taking $p(x) = |\phi(x)|$ for all $x \in E$.

In (E, τ), let Bd, Int, and — denote the boundary, interior, and closure, respectively, with respect to τ. For any subset X of (E, τ), let (X, τ) denote the set X with the relative topology with respect to (E, τ).

Let p be a seminorm on a vector space E. Then (E,p) denotes the seminormed vector space E with the topology determined by p. For a subset $A \subset E$ and an $x \in E$, we denote $d_p(x,A) = \inf\{p(x-y) : y \in A\}$.

For a convex set A, a function $g : A \to E$ is said to be

(i) *almost p-affine* if

$$p(g(rx+(1-r)y)-u) \leq rp(gx-u)+(1-r)p(gy-u);$$

(ii) *almost p-quasiconvex* if

$$p(g(rx+(1-r)y)-u) \leq \max\{p(gx-u), p(gy-u)\}$$

for $x, y \in A$, $u \in E$, and $r \in (0,1)$.

Note that (i) implies (ii), but not conversely. Those concepts were given in Prola [Pr], Hadžić [Hd], Roux and Singh [RS], and Carbone [C1].

A *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. For such spaces, we have the following version of the Allen type variational inequality due to the author [P4, Theorem 0]:

Theorem 0. *Let X be a convex space, $\psi : X \times X \to \overline{\mathbb{R}}$ an extended real function, and K a nonempty compact subset of X. Suppose that*

(0.1) $\psi(x,x) \leq 0$ *for all* $x \in X$*;*

(0.2) *for each* $y \in X$*,* $\{x \in X : \psi(x,y) > 0\}$ *is compactly open;*

(0.3) *for each* $x \in X$*,* $\{y \in X : \psi(x,y) > 0\}$ *is convex or empty; and*

(0.4) *for each* $N \in \langle X \rangle$*, there exists a compact convex subset* L_N *of* X *containing* N *such that, for each* $x \in L_N \backslash K$*, there exists a* $y \in L_N$ *satisfying* $\psi(x,y) > 0$.

Then there exists an $x_0 \in K$ *such that* $\psi(x_0,y) \leq 0$ *for all* $y \in X$*.*

In Theorem 0, $\langle X \rangle$ denotes the set of all nonempty finite subsets of X.

For a subset X of a vector space E and $x \in E$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = \{x + r(u-x) \in E : u \in X,\ r > 0\},$$
$$O_X(x) = \{x + r(u-x) \in E : u \in X,\ r < 0\}.$$

For a topological space X, a function $f : X \to \mathbb{R}$ is said to be *lower semicontinuous* (l.s.c.) if $\{x \in X : fx > r\}$ is open for each $r \in \mathbb{R}$.

RESULTS ON BEST APPROXIMATIONS

The following is a general Fan type best approximation theorem and a variant of the Prolla type:

Theorem 1. *Let X be a convex space, K a nonempty compact subset of X, (E, p) a seminormed vector space containing X as a subset, and $f, g : X \to (E, p)$ continuous maps. Suppose that*

(1.1) *$p(gx - fx) \leq p(x - fx)$ for all $x \in X$; and*

(1.2) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash K$, there exists a $y \in L_N$ satisfying $p(gx - fx) > p(y - fx)$.*

Then there exists an $x_0 \in K$ such that

$$p(gx_0 - fx_0) \leq d_p(fx_0, X).$$

Moreover, if $gx_0 \in X$, then we have

$$p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_X(gx_0)).$$

In this case, $gx_0 \in \operatorname{Bd} X$ and $fx_0 \notin \overline{I}_X(gx_0)$ whenever $p(gx_0 - fx_0) > 0$.

Proof. Define $\psi : X \times X \to \mathbb{R}$ by $\psi(x, y) = p(gx - fx) - p(y - fx)$ for $(x, y) \in X \times X$ and use Theorem 0. Then

(0.1) $\psi(x, x) \leq 0$ for all $x \in X$ by (1.1);

(0.2) for each $y \in X$, $\{x \in X : p(gx - fx) \leq p(y - fx)\}$ is closed since $x \mapsto \psi(x, y)$ is l.s.c.;

(0.3) for each $x \in X$, $\{y \in X : p(gx - fx) > p(y - fx)\}$ is convex; in fact, for any y_1 and y_2 in the set and $0 < r < 1$, we have

$$\begin{aligned} p(ry_1 + (1-r)y_2 - fx) &\leq rp(y_1 - fx) + (1-r)p(y_2 - fx) \\ &< rp(gx - fx) + (1-r)p(gx - fx) \\ &= p(gx - fx); \end{aligned}$$

(0.4) clearly follows from (1.2).

Therefore, by Theorem 0, there exists an $x_0 \in K$ such that $\psi(x_0, z) = p(gx_0 - fx_0) - p(z - fx_0) \leq 0$ for all $z \in X$. This shows that $p(gx_0 - fx_0) \leq d_p(fx_0, X)$.

If $gx_0 \in X$, then for $z = I_X(gx_0)\backslash X$, there exist $u \in X$ and $r > 1$ such that $z = gx_0 + r(u - gx_0)$. Suppose that $p(gx_0 - fx_0) > p(z - fx_0)$. Since

$$u = \frac{1}{r}z + (1 - \frac{1}{r})gx_0 \in X,$$

we have

$$p(u - fx_0) \leq \frac{1}{r}p(z - fx_0) + (1 - \frac{1}{r})p(gx_0 - fx_0) < p(gx_0 - fx_0),$$

a contradiction. Therefore, $p(gx_0 - fx_0) \leq p(z - fx_0)$ for all $z \in I_X(gx_0)$ and hence $p(gx_0 - fx_0) \leq d_p(fx_0, \overline{I}_X(gx_0))$. Since $gx_0 \in \overline{I}_X(gx_0)$, we have $p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_X(gx_0))$.

Suppose that $p(gx_0 - fx_0) > 0$. Then clearly $fx_0 \notin \overline{I}_X(gx_0)$. If $gx_0 \in \operatorname{Int} X$, then $fx_0 \in \overline{I}_X(gx_0) = E$, which leads $d_p(fx_0, \overline{I}_X(gx_0)) = 0$, a contradiction. Therefore, $gx_0 \in \operatorname{Bd} X$. This completes our proof.

Remarks. 1. In certain case, the inward set $I_X(gx_0)$ in the conclusion of Theorem 1 can be replaced by the outward set $O_X(gx_0)$. See Park [P1-3].

2. The continuities of f, g, and p are used only to assure the compactly closedness of $\{x \in X : \psi(x, y) \leq 0\}$ for each $x \in X$ and that $p(gx_0 - fx_0) = d_p(fx_0, I_X(gx_0))$ implies $p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_X(gx_0))$. Therefore, instead of assuming those continuities, it suffices to assume one of the following:

(i) For each $y \in X$, $x \mapsto \psi(x, y)$ is l.s.c.

(ii) f and g are continuous and p is l.s.c.

3. Let (E, τ) be a Hausdorff topological vector space and X a convex subset of E. Note that any $p \in S(E, \tau)$ is continuous on (E, τ) and that the continuity of $f : (X, \tau) \to (E, \tau)$ implies that of $f : (X, \tau) \to (E, p)$. Moreover, every $p \in S(E, \tau)$ is l.s.c. on (E, w), and the continuity of $f : (X, w) \to (E, \tau)$ implies that of $f : (X, w) \to (E, p)$.

Particular Forms. We list some known particular forms of Theorem 1 in chronological order:

1. Ky Fan [F1, Theorem 2]: $X = (X, p) = K$, $g = 1_X$, and p is a norm. This is the origin of Theorem 1.

2. Ky Fan [F2, Theorem 3], [F3, Theorem 7]: $X = (X, p)$, $g = 1_X$, and p is a norm.

3. Kapoor [Ka, Theorem 2]: $X = (X, w) = K$, $g = 1_X$, p is a norm, and $f : (X, w) \to (E, p)$ is (strongly) continuous.

4. Sehgal and Singh [SS1, Theorem 1]: $X = (X, w) = K$, $g = 1_X$, (E, τ) is a locally convex Hausdorff topological vector space, and $f : (X, w) \to (E, \tau)$ is (strongly) continuous.

5. Sehgal, Singh, and Smithson [SSS, Theorem 1]: (E, τ) is a locally convex topologicla vector space, $X = (X, w)$, $p \in S(E, \tau)$, $f : (X, w) \to (E, \tau)$ continuous, and $g = 1_X$.

6. [SSS, Theorem 3]: p is a norm, $f : (X, w) \to (E, p)$ continuous, and $g = 1_X$.

7. Park [P1, Theorem 1.1], [P2, Theorem 1.1$'$]: $X = (X, p)$ and p is a norm.

8. Lin [L, Corollary 1]: p is a norm, $X = (X, w)$, $f : (X, w) \to (E, p)$ continuous, and $g = 1_X$.

9. Sehgal, Singh, and Whitfield [SSW, Theorem 4]: $X = (X, w)$, $g = 1_X$, p is a norm, and $f : (X, w) \to (E, p)$ is (strongly) continuous.

10. Park [P3, Theorem 1]: $X = (X, p)$, $g = 1_X$, and p is a norm.

The following is a general Prolla type best approximation theorem:

Theorem 2. *Let X be a convex space, K a nonempty compact subset of X, (E, p) a seminormed vector space, and $f, g : X \to (E, p)$ continuous maps. Suppose that*

(2.1) *g is almost p-quasiconvex; and*

(2.2) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash K$, there exists a $y \in L_N$ such that $p(gx - fx) > p(gy - fx)$.*

Then there exists an $x_0 \in K$ *such that*

$$p(gx_0 - fx_0) = d_p(fx_0, g(X)).$$

Further, if $g(X)$ *is convex, then*

$$p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_{g(X)}(gx_0)).$$

In this case, $gx_0 \in \operatorname{Bd} g(X)$ *and* $fx_0 \notin \overline{I}_{g(X)}(gx_0)$ *if* $p(gx_0 - fx_0) > 0$.

Proof. Define $\psi : X \times X \to \mathbb{R}$ by $\psi(x, y) = p(gx - fx) - p(gy - fx)$ for $(x, y) \in X \times X$ and use Theorem 0. Then

(0.1) $\psi(x, x) = 0$ for all $x \in X$;

(0.2) for each $y \in X$, $\{x \in X : p(gx - fx) > p(gy - fx)\}$ is open since $x \mapsto \psi(x, y)$ is continuous;

(0.3) for each $x \in X$, $\{y \in X : p(gx - fx) > p(gy - fx)\}$ is convex; in fact, for any y_1 and y_2 in the set and $0 < r < 1$, we have

$$\begin{aligned} p(g(ry_1 + (1 - r)y_2) - fx) &\le \max\{p(gy_1 - fx), p(gy_2 - fx)\} \\ &< p(gx - fx) \end{aligned}$$

by (2.1); and

(0.4) clearly follows from (2.2).

Therefore, by Theorem 0, there exists an $x_0 \in K$ such that $\psi(x_0, y) = p(gx_0 - fx_0) - p(gy - fx_0) \le 0$ for all $y \in X$; that is, $p(gx_0 - fx_0) = d_p(fx_0, g(X))$. Further, if $g(X)$ is convex, as in the proof of Theorem 1, we have $p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_{g(X)}(gx_0))$.

Particular Forms. If X is a subset of E, then we have the following particular forms of Theorem 2.

1. Ky Fan [F1, Theorem 2]: $X = (X,p) = K$, $g = 1_X$, and p is a norm. This is also the origin of Theorem 2.

2. Ky Fan [F2, Theorem 3], [F3, Theorem 7]: $X = (X,p)$, $g = 1_X$, and p is a norm.

3. Prolla [Pr, Theorem]: $X = (X,p) = K$, p is a norm, and g is almost (norm) affine.

4. Hadžić [Hd, Theorem 4]: $X = (X,p)$, p is a norm, and g is almost (norm) affine.

5. Sehgal, Singh, and Smithson [SSS, Theorems 1 and 3]: (E,τ) is a locally convex topological vector space, $X = (X,w)$, $p \in S(E,\tau)$, $f : (X,w) \to (E,\tau)$ continuous, and $g = 1_X$.

6. Sessa [Se, Theorem 2]: p is a norm and g is almost affine.

7. Roux and Singh [RS, Theorem 1]: g is almost p-affine.

8. Park [P3, Theorem 3]: $X = (X,p)$, $g = 1_X$, and p is a norm.

9. Sehgal, Singh, and Gastl [SSG, Corollary 2]: $X = (X,p) = K$ and p is a norm.

10. Carbone [C1, Theorem 2]: $X = (X,p)$ and p is a norm. Note that [C1, Theorem 1] is a simple consequence of [C1, Theorem 2]. Note that [C2] contains some related results, some of which are incorrectly stated.

11. Sessa and Singh [SeS, Theorem 7 and Corollary 3]: $X = (X,w)$, p is a norm, $f : (X,w) \to (E, \|\cdot\|)$ and $g : (X,w) \to (E,w)$ are sequentially continuous, and g is almost (norm) affine.

If g in Theorem 2 satisfies

(2.1)$'$ $g([x,y]) = [gx, gy]$ for $x,y \in X$, where $[x,y] = \{rx + (1-r)y : 0 \le r \le 1\}$; or equivalently,

(2.1)$''$ $g^{-1}([u,v])$ is convex for $u,v \in g(X)$,

then (2.1) holds automatically and $g(X)$ is convex. Therefore, we have the following particular forms of Theorem 2:

12. Lin [L, Theorem 2]: (X, w) is a convex subset of a locally convex Hausdorff topological vector space, p is a norm, and g satisfies $(2.1)''$.

13. Sessa and Singh [SeS, Theorem 6]: (X, w) is a convex subset of E, p is a norm, and $f : (X, w) \to (E, p)$ is sequentially strongly continuous, and $g : (X, w) \to (E, w)$ is sequentially weakly continuous and satisfies $(2.1)''$.

The conclusions of Theorems 1 and 2 have some equivalent statements as follows:

Theorem 3. *Under the hypothesis of Theorem 1, the following are equivalent:*

(i) *There is an $x_0 \in K$ such that*

$$p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_X(gx_0)).$$

(ii) *If $h : X \to (E, p)$ is a function such that, for any $x \in K$ with $gx \neq hx$, there exists a $y \in \overline{I}_X(gx)$ satisfying*

$$p(gx - fx) > p(y - fx),$$

then $gx_0 = hx_0$ for some $x_0 \in K$.

(iii) *If $h : X \to (E, p)$ is a function such that $hx \in \overline{I}_X(gx)$ for all $x \in X$ and*

$$p(gx - fx) > p(hx - fx)$$

for all $x \in K$ with $gx \neq hx$, then $gx_0 = hx_0$ for some $x_0 \in K$.

Proof. (i) $\Longrightarrow$ (ii) For the $x_0 \in K$ in (i), suppose that $gx_0 \neq hx_0$. Then there exists a $y \in \overline{I}_X(gx_0)$ such that $p(gx_0 - fx_0) > p(y - fx_0)$. This contradicts (i).

(ii) $\Longrightarrow$ (iii) Put $y = hx$.

(iii) $\Longrightarrow$ (i) Suppose that for any $x \in K$, there exists a $y \in \overline{I}_X(gx)$ such that $p(gx - fx) > p(y - fx)$. Note that, by the hypothesis of Theorem 1, for each $x \in X \backslash K$, we have $x \in L_{\{x\}} \backslash K$ and hence, there exists a $y \in L_{\{x\}} \subset X \subset \overline{I}_X(gx)$ satisfying $p(gx - fx) > p(y - fx)$. Now, for any $x \in X$, choose hx to be one of such $y \in \overline{I}_X(gx)$. Then $h : X \to (E, p)$ satisfies $hx \in \overline{I}_X(gx)$ for each $x \in X$. Note that h and g have no coincidence point in K. This contradicts (iii). Therefore, there is an $x_0 \in K$ such that $p(gx_0 - fx_0) \leq p(z - fx_0)$ for all $z \in \overline{I}_X(gx_0)$.

This completes our proof.

Remark. Similarly, under the hypothesis of Theorem 2, if $g(X)$ is convex, then Theorem 3 holds for $\overline{I}_{g(X)}(gx_0)$ instead of $\overline{I}_X(gx_0)$.

Particular Forms.

1. Browder [B2, Corollaries 1 and 2]: $X = (X, p) = K$, $g = 1_X$, $f = h$, and p is a norm in (ii).
2. Park [P1, Theorem 1.2]: $X = (X, p)$ and p is a norm.
3. Park [P1, Theorem 2.2]: $X = (X, p) = K$ and p is a norm.

Moreover, Theorems 1 or 2 with $g = 1_X$ implies the following approximation result:

Theorem 4. *Let X be a convex space, K a nonempty compact subset of X, and (E, p) a seminormed vector space containing X as a subset such that $1_X : X \to (X, p)$ is continuous. Let $z \in E$. Suppose that, for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash K$, there exists a $y \in L_N$ satisfying $p(x - z) > p(y - z)$. Then there exists a $u \in K$ such that*

$$p(u - z) = d_p(z, X).$$

Proof. Use Theorems 1 or 2 by letting $g = 1_X$ and $fx = z$ for all $x \in X$.

Particular Forms.

1. For a locally convex Hausdorff topological vector space E and a weakly compact convex subset $X = K$, Theorem 4 reduces to Sehgal and Singh [SS1, Corollary 3].
2. For a convex subset X of a normed vector space (E, p), Theorem 4 extends Park [P1, Corollary].

COINCIDENCE RESULTS

The following is a basis for the existence of coincidence points of maps in a normed vector space:

Theorem 5. *Under the hypothesis of Theorem 1, if $g(X) \subset X$, then there exists an $x_0 \in K$ such that $p(gx_0 - fx_0) = 0$ whenever one of the following conditions holds:*

(a) *For any $x \in K$ with $p(gx - fx) > 0$ and $gx \in \operatorname{Bd} X$, there exists a $y \in \overline{I}_X(gx)$ satisfying*

$$p(gx - fx) > p(y - fx).$$

(b) *For any $x \in K$ with $p(gx - fx) > 0$ and $gx \in \operatorname{Bd} X$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that*

$$|\lambda| < 1 \text{ and } y = \lambda gx + (1 - \lambda) fx \in \overline{I}_X(gx).$$

(c) *For any $x \in K$ with $gx \in \operatorname{Bd} X$, we have $fx \in \overline{I}_X(gx)$.*

Proof. (a) Suppose that for any $x \in K$, we have $p(gx - fx) > 0$. If $gx \in \operatorname{Int} X$, the point $y = (gx + fx)/2 \in \overline{I}_X(gx) = E$ satisfies $p(gx - fx) > p(y - fx)$. Therefore, from (a), for any $x \in K$, there exists a $y \in \overline{I}_X(gx)$ satisfying $p(gx - fx) > p(y - fx)$. This contradicts the conclusion of Theorem 1.

(b) We show (b) $\Longrightarrow$ (a). In fact, we have

$$p(y - fx) = |\lambda| p(gx - fx) < p(gx - fx).$$

(c) Put $\lambda = 0$ in (b).

Remark. Similarly, under the hypothesis of Theorem 2, if $g(X)$ is convex, then Theorem 5 holds for $\overline{I}_{g(X)}(gx)$ instead of $\overline{I}_X(gx)$.

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Particular Forms.

1. If p is a norm, then Theorem 5 is a coincidence theorem. Moreover, if $g = 1_X$, then Theorem 5 includes earlier generalizations of the Schauder fixed point theorem due to Halpern and Bergman [HB], Fan [F1], Browder [B2], and Reich [R2].

2. If p is a norm, then Theorem 5 includes Park [P1, Theorems 1.3 and 2.3], [P3, Corollaries 1.1-1.4 and 2.1-2.4], and Theorem 5 under the hypothesis of Theorem 2 extends Lin [L, Theorem 3].

3. Theorem 5(b) extends Roux and Singh [RS, Theorems 3 and 4] and Hadžić [Hd, Theorem 3].

4. If the closed ball $B(0,r)$ with center 0 and radius $r > 0$ in a normed vector space E is weakly compact (for example, E is a reflexive Banach space) and $f : B(0,r) \to (E, \|\cdot\|)$ is continuous, then the following boundary conditions are particular to Theorem 5(c) with $g = 1_X$:

(i) if $fx = \alpha x$ for $x \in \mathrm{Bd}\, B(0,r)$, then $\alpha < 1$.

(ii) $\|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2$ for $x \in \mathrm{Bd}\, B(0,r)$.

Therefore, [SSW, Corollary 2 and its consequences] follow from Theorem 5(c).

From Theorem 1, we have the following coincidence theorem for a topological vector space having sufficiently many linear functionals:

Theorem 6. *Let X be a convex space, K a nonempty compact subset of X, and (E,τ) a topological vector space on which E^* separates points such that E contains X as a subset. Let $f, g : X \to (E,w)$ be continuous maps satisfying* (1.1) *and* (1.2) *for each $p \in S(E,w)$ and $g(X) \subset X$. Then either*

(i) *f and g have a coincidence point $x_0 \in K$, or*

(ii) *there exist a $q \in S(E,w)$ and an $x_0 \in K$ such that $gx_0 \in BdX$ and*

$$0 < q(gx_0 - fx_0) = d_q(fx_0, \overline{I}_X(gx_0)).$$

Proof. For any $p \in S(E,w)$, f and g can be regarded as $f : X \to (E,p)$ and $g : X \to (E,p)$. Then the conclusion of Theorem 1 holds. Let

$$F[p] = \{x \in K : p(gx - fx) = 0\}.$$

Suppose that $F[q] = \emptyset$ for some $q \in S(E,w)$. Then, from Theorem 1, we obtain conclusion (ii). Suppose that $F[p] \neq \emptyset$ for any $p \in S(E,w)$. Then $F[p]$ is closed in K as $x \mapsto p(gx - fx)$ is continuous on K. For any $\{p_1, \cdots, p_n\} \in \langle S(E,w) \rangle$, we have $\sum_{i=1}^n p_i \in S(E,w)$ and $F[\sum_{i=1}^n p_i] \subset \bigcap_{i=1}^n F[p_i]$. Therefore, $\{F[p] : p \in S(E,w)\}$ has the finite intersection property. Since K is compact, there exists an $x_0 \in \bigcap \{F[p] : p \in S(E,w)\}$. Therefore, $p(gx_0 - fx_0) = 0$ for all $p \in S(E,w)$. Since E^* separates points of E, we should have $gx_0 = fx_0$. This completes our proof.

From Theorem 2, we have the following:

Theorem 7. *Let X be a convex space, K a nonempty compact subset of X, and (E,τ) a topological vector space on which E^* separates points. Let $f, g : X \to (E,w)$ be continuous maps satisfying* (2.1) *and* (2.2) *for each $p \in S(E,w)$ and $g(X)$ is convex. Then the conclusion of Theorem 6 holds for $\overline{I}_{g(X)}(gx_0)$ instead of $\overline{I}_X(gx_0)$.*

Proof. Follow that of Theorem 6 using Theorem 2 instead of Theorem 1.

Remarks. 1. In Theorems 6 and 7, the sets K and L_N may depend on $p \in S(E,w)$. However, for the simplicity we fixed them.

2. In Theorem 6, the continuities of $f, g : X \to (E,w)$ can be replaced by those of $f, g : X \to E$ where E has any topology finer than the weak topology and X, as a convex space, has any topology finer than the relative weak topology.

Particular Forms. We list some particular forms of Theorems 6 and 7.

1. For a locally convex Hausdorff topological vector space E, $X = K$, and $g = 1_X$, Theorems 6 and 7 improve Fan [F1, Theorem 1], Reich [R3, Theorem 1], and Sehgal and Singh [SS1, Corollary 1].

2. For a locally convex Hausdorff topological vector space E, Theorem 6 generalizes Park [P2, Theorem 1.1] and Kim [Ki, Theorem 3.1].

3. The second form of Ha [H, Theorem 3] follows from Theorem 7.

In view of Theorems 6 and 7, for a topological vector space having sufficiently many linear functionals, Theorem 5 reduces to the following:

Theorem 8. *Let X be a convex space, K a nonempty compact subset of X, and (E, τ) a topological vector space on which E^* separates points such that E contains X as a subset. Let $f, g : X \to (E, w)$ be continuous maps satisfying $g(X) = X$ and either*

(A) (1.1) *and* (1.2) *for each $p \in S(E, w)$; or*

(B) (2.1)*, and* (2.2) *for each $p \in S(E, w)$.*

Then f and g have a coincidence point if one of (a)*,* (b)*, and* (c) *of Theorem 5 holds for each $p \in S(E, w)$.*

Proof. Note that each of (a), (b), and (c) violates the conclusion (ii) of Theorems 6 and 7, as we have seen in the proof of Theorem 5.

Remark. Without assuming $g(X) = X$, Theorem 8(B) holds if $g(X)$ is convex and we replace $\overline{I}_X(gx)$ by $\overline{I}_{g(X)}(gx)$ in the conditions (a), (b), and (c) of Theorem 5.

Particular Forms. For $X = K$ and $g = 1_X$, if $fx \in \overline{I}_X(x)$ for all $x \in X$, then Theorem 8 unifies Roux and Singh [RS, Theorems 5 and 6], which generalizes earlier works of Browder [B1], Fan [F1], Reich [R1,3], and Kaczynski [K] on generalizations of the Tychonoff fixed point theorem. Theorem 8(A) extends Park [P2, Theorem 3.1] and Kim [Ki, Theorem 3.4]. Finally, Theorem 8(B) extends Ha [H, Theorem 4] and Hadžić [Hd, Theorem 5].

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