APPLICATIONS OF MAXIMIZABLE LINEAR FUNCTIONALS ON CONVEX SETS

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Abstract

Recently, the author and J.S.Bae [**14**] removed the paracompactness assumption in Bellenger's theorem [**3**] on the existence of maximizable quasiconcave functions on convex spaces. Fan [**8**] and Simons [**20**] applied particular forms of this theorem to various problems. In this paper, we give some other applications of the existence theorem to the Fan type nonseparation theorems and the existence of maximizable linear functionals having certain preassigned properties.

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1. Introduction

Recently, the author and J.S.Bae [**14**, Theorem 1] removed the paracompactness assumption in Bellenger [**3**, Theorem 1] on the existence of maximizable quasiconcave functions on convex spaces. This existence theorem extends and unifies earlier works of Fan [**8**, Theorem 8] and Simons [**20**, Theorem 0.1].

Fan [**8**] used his theorem to obtain coincidence and fixed point theorems, matching theorems for closed coverings of convex sets, and a generalization of the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem. Recently, the author and S.K.Kim [**15**] applied Fan [**8**, Theorem 8] to obtain generalized extremal principles originated from Mazur and Schauder.

On the other hand, Simons [**18**], [**19**], [**20**] and Bellenger and Simons [**4**] used [**20**, Theorem 0.1] to obtain generalized Kakutani type fixed point theorems, certain "approximation" theorems, fixed point theorems for noncontinuous multifunctions, and existence of zeros of multifunctions. In our works [**12**], [**13**], our version of Bellenger's theorem or some of its consequences are used to obtain new generalized results on coincidences or fixed points, surjectivity, existence of critical points (or zeros), matching theorems, and many others.

In the present paper, we give some other applications of the existence theorem of maximizable quasiconcave functions on convex spaces to the Ky Fan type nonseparation theorems and the existence of maximizable linear functionals with certain properties.

Section 2 deals with preliminaries. Theorem 1 is a particular form of our version of Bellenger's theorem; that is, the existence theorem of maximizable linear functionals on convex sets in a topological vector space.

In Section 3, we obtain Theorem 3, a far-reaching generalization of the Ky Fan type nonseparation theorems due to Fan [**6**], [**7**], Reich [**16**], Takahashi [**21**], and Lee and Tan [**11**]. Those are bases of generalizations of the Brouwer or Kakutani type coincidence theorems for upper demicontinuous multifunctions defined on convex subsets of a topological vector space.

Section 4 deals with noncompact version of Simons [**18**, Remark 4.6] on the existence of the maximizable continuous linear functionals having certain preassigned properties. In fact, all of the results in Section 4 are variations of Theorem 1 and generalized versions of known results due to Fan [**6**], Browder [**5**], Takahashi [**22**], and Simons [**18**].

2. Existence of maximizable linear functionals

A real Hausdorff topological vector space will be abbreviated to a t.v.s. For a t.v.s. E, E^* denotes the set of all continuous linear functionals.

A *convex space* X is a nonempty convex set X (in a vector space) with any topology that induces the Euclidean topology on the closed convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$, there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. See Lassonde [**10**]. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$, where $x \in X$.

Recall that an extended real-valued function g defined on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : gx > r\}$ [resp. $\{x \in X : gx < r\}$] is open for each real r.

An open half-space H in a t.v.s. E is a set of the form $H = \{x \in E : fx > t\}$ for some nontrivial $f \in E^*$ and some real t.

Let X be a topological space, E a t.v.s., and $F : X \to 2^E$. Then

(i) F is *upper semi-continuous* (u.s.c.) if for each $x \in X$ and each open set U in E containing Fx, there exists an open neighborhood N of x in X such that $F(N) \subset U$;

(ii) F is *upper demi-continuous* (u.d.c.) if for each $x \in X$ and each open half-space H in E containing Fx, there exists an open neighborhood N of x in X such that $F(N) \subset H$ [**6**], [**7**] ; and

(iii) F is *upper hemi-continuous* (u.h.c.) if for each $f \in E^*$ and each real r, the set $\{x \in X : \sup f(Fx) < r\}$ is open in X; i.e., the function $\sup fF : X \to \mathbf{R} \cup \{+\infty\}$ is u.s.c.; equivalently, for each $f \in E^*$ and each real r, the set $\{x \in X : \inf f(Fx) > r\}$ is open in X; i.e., $\inf fF : X \to \mathbf{R} \cup \{-\infty\}$ is l.s.c. [**1**], [**2**].

Note that u.s.c. $\Longrightarrow$ u.d.c. $\Longrightarrow$ u.h.c. and that if the multifunction F is compact-valued, then u.d.c. $\Longleftrightarrow$ u.h.c. If $F, G : X \to 2^E$ are u.h.c., so is $F + G$.

The following is a simple consequence of the existence theorem for maximizable quasiconcave functions on convex spaces due to the author and Bae [**14**, Theorem 1].

Theorem 1. *Let X be a convex space and E a t.v.s. containing X as a subset.*

(1.0) *for each $f \in E^*$, $f|_X$ is continuous on X ;*

(1.1) *for each $x \in X$, Sx is a nonempty convex subset of E^* ;*

(1.2) *for each $f \in E^*$, $S^{-1}f$ is compactly open in X ; and*

(1.3) *there exist a c-compact set $L \subset X$ and a nonempty compact set $K \subset X$ such that for every $x \in X \backslash K$ and $f \in Sx$, we have $fx < \max f[x, L]$.*

Then there exist an $\hat{x} \in K$ and an $f \in S\hat{x}$ such that $f\hat{x} = \max f(X)$.

As in our previous work [**13**], we obtain the following :

Theorem 2. *Let X, E, L, and K be as in Theorem 1, and $P, Q : X \to 2^E \backslash \{\emptyset\}$ multifunctions. Suppose that, for each $f \in E^*$,*

(2.0) *$f|_X$ is continuous on X ;*

(2.1) *$X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ is compactly closed in X ;*

(2.2) *for each $x \in K$, $fx = \max f(X)$ implies $x \in X_f$; and*

(2.3) *for each $x \in X \backslash K$, $fx = \max f[x, L]$ implies $x \in X_f$.*

Then there exists an $x \in \bigcap\{X_f : f \in E^*\}$.

In Theorems 1 and 2, we do not require any concrete connection between topologies of X and E except (1.0). Therefore, we may assume that

(i) as a convex space, X has any topology finer than the relative weak topology with respect to E, and

(ii) E has a topology finer than its weak topology.

Note that Theorem 2 is a base for various coincidence and fixed point theorems, and surjectivity results on u.h.c. multifunctions defined on non-compact convex set in our previous works [**12**], [**13**]. However, Theorem 2 does not include the Ky Fan type nonseparation theorems in [**6**], [**7**], which were used to obtain fixed point results on u.d.c. multifunctions. In the next section, we obtain a very general nonseparation theorem.

3. The Ky Fan type nonseparation theorems

Two subsets M, N of a t.v.s. E are said to be *strictly separated by a closed hyperplane* if there exist an $f \in E^*$ and a $t \in \mathbf{R}$ such that $M \subset \{x \in E : fx > t\}$ and $N \subset \{x \in E : fx < t\}$.

As a direct application of Theorem 1, we obtain the following Ky Fan type nonseparation theorem.

Theorem 3. *Let X be a convex space, L a c-compact subset, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and $P, Q : X \to 2^E \setminus \{\emptyset\}$. Suppose that for each $f \in E^*$,*

(3.0) *$f|_X$ is continuous on X ;*

(3.1) *$X'_f = \{x \in X :$ there exists no $r \in \mathbf{R}$ such that $f(Px) \subset (-\infty, r)$ and $f(Qx) \subset (r, \infty)\}$ is compactly closed in X ;*

(3.2) *for each $x \in K$, $fx = \max f(X)$ implies $x \in X'_f$; and*

(3.3) *for each $x \in X \setminus K$, $fx = \max f[x, L]$ implies $x \in X'_f$.*

Then there exists an $x \in X$ such that Px and Qx can not be strictly separated by a closed hyperplane in E.

Proof. Suppose that for each $x \in X$, Px and Qx can be strictly separated by a closed hyperplane. Define $S : X \to 2^{E^*}$ by

$$Sx = \{f \in E^* : f(Px) \subset (-\infty, r) \text{ and } f(Qx) \subset (r, \infty) \text{ for some } r \in \mathbf{R}\}$$

for each $x \in X$. Then Sx is nonempty and convex. By (3.1), $S^{-1}f = X \setminus X'_f$ is compactly open for each $f \in E^*$. Note that (3.3) implies (1.3) in Theorem 1. Therefore, by Theorem 1, there exist an $\hat{x} \in K$ and an $f \in S\hat{x}$ such that $f\hat{x} = \max f(X)$. Note that $\hat{x} \in S^{-1}f = X \setminus X'_f$, which contradicts (3.2). This completes our proof.

Condition (3.1) is a certain "continuity" condition. In fact, let us consider

(3.1)$'$ P and Q are u.d.c.

Note that (3.1)$'$ $\Longrightarrow$ (3.1), but not conversely.

For example, let $X = [0, 1]$ in $E = \mathbf{R}$, $P = 1_X$, $Qx = \{x\}$ for $x \in X \setminus (1/3,\ 2/3)$ and $Qx = \{1\}$ for $x \in (1/3,\ 2/3)$. Then (3.1) holds, but Q is not continuous; that is, not u.d.c.

Note also that $X'_f \subset X_f$, and hence (3.2) $\Longrightarrow$ (2.2) and (3.3) $\Longrightarrow$ (2.3). Therefore, if we assume $(3.1)'$ in Theorem 3, then we obtain the conclusion of Theorem 2 as follows :

($*$) There exists an $x \in X$ such that $x \in \bigcap\{X_f : f \in E^*\}$.

Since the conclusion of Theorem 3 is more concrete than ($*$), Theorem 3 does not follow from Theorem 2.

Condition (3.2) is a generalized "boundary" condition. In fact, (3.2) is equivalent to

$(3.2)'$ for each $x \in K \cap \operatorname{Bd} X$, $fx = \max f(X)$ implies $x \in X'_f$, where Bd denotes the boundary w.r.t. E.

This can be shown as in Fan [**8**, p.528]. Moreover, if $P = 1_X$, then (3.2) is related to certain whereabouts of values of Q on $\operatorname{Bd} X$. For details, see [**12**].

Some particular forms of (3.2) for $X = K$ have appeared in literature as follows :

(3.2.1) (Fan [**6**, Theorem 8]) For each $x \in \delta(X)$, $I_{Px}(x) \cap X \neq \emptyset$ and $O_{Qx}(x) \cap X \neq \emptyset$.

(3.2.2) (Fan [**6**, Theorem 5]) For each $x \in \delta(X)$, $Px \cap I_X(Qx) \neq \emptyset$ $[Px \cap O_X(Qx) \neq \emptyset]$.

(3.2.3) (Fan [**7**, Theorem 3]) For each $x \in X$ and each $f \in E^*$ such that $fx = \max f(X)$, there exists $u \in Px$ and $v \in Qx$ such that $fu \geq fv$.

(3.2.4) (Reich [**16**, Proposition 2.1]) Qx is compact and $Px \cap \overline{I_X(Qx)} \neq \emptyset$ $[Px \cap \overline{O_X(Qx)} \neq \emptyset]$ for all $x \in X$.

(3.2.5) (Takahashi [**21**, Theorem 8]) For any $x \in X$ and $f \in E^*$, either $f(x-u) \leq f(x-v)$ for all $u \in Px$ and $v \in Qx$ or $0 > \inf_{y \in X} f(x-y)$.

(3.2.6) (Lee and Tan [**11**, Theorem 3]) For each $x \in \delta(X)$, $Px \cap \overline{I_X(x)} \neq \emptyset$ and $Qx \cap \overline{O_X(x)} \neq \emptyset$.

(3.2.7) (Lee and Tan [**11**, Theorem 1]) For each $x \in X$, there exist two points $u \in Px$, $v \in Qx$, two nets $u_\alpha,\ v_\alpha$ in E, and a net y_α in X such that $u_\alpha \to u$, $v_\alpha \to v$ and $u_\alpha - v_\alpha = \lambda_\alpha(x - y_\alpha)$ for some net λ_α in $(0, \infty)$.

Here, the algebraic boundary $\delta(X)$ of X [**6**] is defined by

$$\delta(X) = \{y \in X : \text{there exists } z \in K \text{ such that } y + \lambda z \notin X \text{ for all } \lambda > 0\}.$$

Note also that

$$I_A(x) = \{x + \lambda(y - x) : y \in A,\ \lambda \geq 0\},$$
$$O_A(x) = \{x + \lambda(y - x) : y \in A,\ \lambda \leq 0\}$$

for $A \subset E$ and $x \in E$,

$$I_X(Qx) = \{v + a(y - x) : v \in Qx,\ y \in X,\ a \geq 0\},$$
$$O_X(Qx) = \{v + a(y - x) : v \in Qx,\ y \in X,\ a \leq 0\},$$

and — denotes the closure w.r.t. E.

It is shown that (3.2.1) $\Longrightarrow$ (3.2.2) in [**6**] and (3.2.2) $\Longrightarrow$ (3.2.3) in [**7**]. If Qx is compact, (3.2.2) $\Longrightarrow$ (3.2.4) [**16**] and (3.2.4) $\Longrightarrow$ (3.2.7) [**11**]. It is also shown that (3.2.6) $\Longrightarrow$ (3.2.7) in [**11**]. It is easy to check (3.2.3) $\Longrightarrow$ (3.2), (3.2.5) $\Longrightarrow$ (3.2), and (3.2.7) $\Longrightarrow$ (3.2).

Finally, Condition (3.3) is a "coercivity" or "compactness" condition. For $X = K$, (3.3) is automatically satisfied and Theorem 3 reduces to Simons [**18**, Remark 4.6, 1st Statement].

Therefore, we can state the following generalization of the above mentioned results (3.2.1)-(3.2.7).

Corollary 3.1. (Simons [**18**]) *Let X be a nonempty compact convex subset of a t.v.s. E, and $P, Q : X \to 2^E \backslash \{\emptyset\}$ u.d.c. multifunctions. If (3.2) holds, then there exists an $x \in X$ such that Px and Qx can not be strictly separated by a closed hyperplane.*

Note that all coincidence or fixed point theorems on u.d.c. multifunctions in Ky Fan [**7**], [**8**] and Granas and Liu [**9**, Theorems 10.1–10.5] are consequences of Theorem 3 or Corollary 3.1. However, those results are already generalized extensively by applying Theorem 2 in our previous work [**13**]. Only one exception not treated in [**13**] can be extended as follows :

Theorem 4. *If, in addition to the hypothesis of Theorem 3, for each $x \in X$, Px and Qx are open convex subsets of E, then P and Q have a coincidence point $x \in X$, that is, $Px \cap Qx \neq \emptyset$.*

Proof. Suppose $Px \cap Qx = \emptyset$ for all $x \in X$. Recall the well-known fact that two disjoint open convex sets in a t.v.s. can be strictly separated by a closed hyperplane. This contradicts Theorem 3.

Note that, for $X = K$ and u.d.c. P and Q, Theorem 4 reduces to Fan [**7**, Theorem 4]. Note also that Theorem 4 may not follow from Theorem 2.

4. Maximizable linear functionals with certain properties

In this section, we consider existence problems on maximizable linear functionals with certain preassigned properties. Our results generalize earlier works of Simons [**18**], Fan [**6**], Browder [**5**], and Takahashi [**22**] .

The following is a contrapositive of Theorem 2.

Theorem 5. *Let X, L, K, E, and P, Q be as in Theorem 3. Suppose that*

(5.0) *for each $f \in E^*$, $f|_X$ is continuous on X ;*

(5.1) *for each $f \in E^*$, $X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ is compactly closed ;*

(5.2) *for each $x \in X$, there exists an $f \in E^*$ such that $\sup f(Px) < \inf f(Qx)$; and*

(5.3) *for each $x \in X \backslash K$, $fx = \max f[x, L]$ implies $x \in X_f$.*

Then there exists an $x \in K$ and an $f \in E^*$ such that $fx = \max f(X)$ and $\sup f(Px) < \inf f(Qx)$.

Proof. Define $S : X \to 2^E$ by

$$Sx = \{f \in E^* : \sup f(Px) < \inf f(Qx)\}$$

for each $x \in X$. Then each Sx is nonempty by (5.2), and convex. For each $f \in E^*$, $S^{-1}f = X \backslash X_f$ is compactly open by (5.1). Also (5.3) implies (1.3). Therefore, by Theorem 1, the conclusion follows.

For the case $X = K$, if P and Q are u.h.c. (dual-u.s.c. in [**18**]), Theorem 5 reduces to Simons [**18**, Remark 4.6, 2nd Statement], which, in turn, generalizes Fan [**6**, Theorem 6].

Corollary 5.1. *Let X, L, K, E and P, Q be as in Theorem 3. Suppose that* (5.0), (5.1), (5.3)*, and the following hold.*

(5.2)$'$ *for each $x \in X$, Px and Qx are convex, and $\overline{Px} \cap \overline{Qx} = \emptyset$.*

If either

(A) *E^* separates points of E and, for each $x \in X$, $\overline{Px}$ and $\overline{Qx}$ are compact, or*

(B) *E is locally convex and, for each $x \in X$, one of $\overline{Px}$ and $\overline{Qx}$ is compact,* *then there exist an* $x \in K$ *and an* $f \in E^*$ *such that* $fx = \max f(X)$ *and* $\sup f(Px) < \inf f(Qx)$.

Proof. By the standard separation theorems in a t.v.s. [**17**], (5.2)$'$ with (A) or (B) implies (5.2). Therefore, by Theorem 5, we have the conclusion.

For the case $X = K$, if P and Q are u.h.c., then Corollary 5.1(B) reduces to Simons [**18**, Remark 4.6, 3rd Statement].

If $P = 1_X$ in Corollary 5.1, then we have

Corollary 5.2. *Let* X, L, K, E, *and* Q *be as in Theorem 3. Suppose that*

(5.0) *for each* $f \in E^*$, $f|_X$ *is continuous on* X *;*

(5.1)$'$ *for each* $f \in E^*$, $X_f = \{x \in X : fx \geq \inf f(Qx)\}$ *is compactly closed ;*

(5.2)$''$ *for each* $x \in X$, Qx *is convex and* $x \notin \overline{Qx}$*; and*

(5.3)$'$ *for each* $x \in X \backslash K$, $fx = \max f[x, L]$ *implies* $x \in X_f$.

If either

(A) E^* *separates points of* E *and* $\overline{Qx}$ *is compact for each* $x \in X$ *; or*

(B) E *is locally convex,*

then there exists an $x \in K$ *and an* $f \in E^*$ *such that*

$$fx = \max f(X) < \inf f(Qx).$$

For the case $X = K$, if Q is u.h.c., then Corollary 5.2(B) reduces to Simons [**18**, Remark 4.6, 4th Statement], which, in turn, generalizes Browder [**5**, Theorem 8] and Takahashi [**22**, Theorem 11].

We conclude this paper with metric analogues of two of the above results in the case when E is normed.

Theorem 6. *Let X be a nonempty convex subset of a normed vector space E, L a c-compact subset, K a nonempty compact subset of X, $h : X \to \mathbf{R}$ an u.s.c. function, and $P, Q : X \to 2^E \backslash \{\emptyset\}$. Suppose that*

(6.1) *for each $f \in E^*$ with $\|f\| \leq 1$, $X_f = \{x \in X : \sup f(Px) + hx \geq \inf f(Qx)\}$ is compactly closed ;*

(6.2) *for each $x \in X$, Px and Qx are convex, and* $\operatorname{dist}(Px, Qx) > hx$ *; and*

(6.3) *for each $x \in X \backslash K$ and each $f \in E^*$ with $\|f\| \leq 1$, $fx = \max f[x, L]$ implies $x \in X_f$.*

Then there exist an $x \in K$ and an $f \in E^*$ such that $\|f\| \leq 1$, $fx = \max f(X)$, and $\sup f(Px) + hx < \inf f(Qx)$.

Proof. We use Theorem 1. Note that (1.0) holds since X has the relative topology with respect to E. Define $S : X \to 2^{E^*}$ by

$$Sx = \{f \in E^* : \|f\| \leq 1,\ \sup f(Px) + hx < \inf f(Qx)\}$$

for $x \in X$. Then each Sx is convex, and nonempty by (6.2). Further, for each $f \in E^*$, if $\|f\| > 1$, then $S^{-1}f$ is empty, and if $\|f\| \leq 1$, then

$$S^{-1}f = \{x \in E^* : \sup\ f(Px) + hx \leq \inf\ f(Qx)\}.$$

In any case, $S^{-1}f$ is compactly open. Since (6.3) implies (1.3), all of the requirements of Theorem 1 are satisfied. Therefore we have the conclusion.

For the case $X = K$, if P and Q are u.h.c., then Theorem 6 reduces to Simons [**18**, Remark 4.6, 5th Statement]. For $P = 1_X$, we have the following :

Corollary 6.1. *Let X, E, L, K, h, and Q be as in Theorem 6. Suppose that*

(6.1)′ *for each $f \in E^*$ with $\|f\| \le 1$, $X_f = \{x \in X : fx + hx \ge \inf f(Qx)\}$ is compactly closed ;*

(6.2)′ *for each $x \in X$, Qx is convex and* dist $(x, Qx) > hx$ *; and*

(6.3) *for each $x \in X \backslash K$ and each $f \in E^*$ with $\|f\| \le 1$, $fx = \max f[x, L]$ implies $x \in X_f$.*

Then there exist an $x \in K$ and an $f \in E^*$ such that $\|f\| \le 1$ and $fx = \max f(X) < \inf f(Qx) - hx$.

For the case $X = K$, if Q is u.h.c., then Corollary 6.1 reduces to Simons [**18**, Remark 4.6, 6th Statement], which, in turn, generalizes Browder [**5**, Theorem 9].

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