ADMISSIBLE CLASSES OF MULTIMAPS ON GENERALIZED CONVEX SPACES

SEHIE PARK[A] AND HOONJOO KIM[B]

Abstract. Recently the first author introduced certain general classes of upper semicontinuous multimaps defined on convex spaces which were shown to be adequate to establish theories on fixed points, coincidence points, KKM maps, variational inequalities, best approximations, and many others. Later we found that, in certain cases, the convex spaces can be replaced by new classes of more general spaces. In this paper we collect examples of such classes of multimaps and generalized convex spaces. Some fundamental properties of such examples are also discussed.

1. Introduction

Recently the first author introduced certain general classes of upper semicontinuous multimaps defined on convex spaces which were shown to be adequate to establish theories on fixed points, coincidence points, KKM maps, variational inequalities, best approximations, and many others. See [50, 54, 55, 56],

Our admissible classes of multimaps (maps) include composites of important maps which appear in nonlinear analysis or algebraic topology. Examples of such maps are continuous functions, Kakutani maps [33], acyclic maps [21, 43, 44, 45, 49, 52, 53], Fan-Browder type maps [22, 15], admissible maps in the sense of Górniewicz [23], permissible maps of Dzedzej [20], approachable maps [7, 8, 9], and many others.

Later we found that, in certain cases, the convex spaces can be replaced by new classes of more general spaces. Actually, our new concept of generalized convex spaces is a generalization of the usual convexity in a topological vector space, Michael's convex structure [38], Pasicki's S-contractible spaces [59, 60, 61], Komiya's convex spaces [34], Lassonde's convex spaces [35], Horvath's pseudoconvex spaces [29] and c-structure [30] or H-spaces [1, 2, 3], Bielawski's simplicial convexity [12], Joó's pseudoconvexity [31], and many others. Those general convexities were developed in connection mainly with the fixed point theory and the KKM theory.

In this paper we investigate fundamental properties of many examples of such classes of multimaps and generalized convex spaces.

2. Preliminaries

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-}(y) := \{x \in X | \, y \in F(x)\}$ for $y \in Y$. As usual, the set $\{(x, y) : y \in F(x)\}$ is called the *graph* of F, or, simply, F. Therefore $(x, y) \in F$ if and only if $y \in F(x)$.

For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$. For any $B \subset Y$, the *lower inverse* of B under F is defined by

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

The *(lower) inverse* of $F : X \multimap Y$ is the map $F^- : Y \multimap X$ defined by $x \in F^-(y)$ if and only if $y \in F(x)$. Given two maps $F : X \multimap Y$ and $G : Y \multimap Z$, the composite $GF : X \multimap Y$ is defined by $(GF)(x) = G(F(x))$ for $x \in X$.

For topological spaces X and Y, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if F is both u.s.c. and l.s.c.

Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact.

Int denotes the interior. For a nonempty set X, $\langle X\rangle$ denote the set of all nonempty finite subsets of a set X.

Let X be a convex space in the sense of Lassonde [35] and Y a topological space. We say that F is *demi-closed* in $X \times Y$ if for each $N \in \langle X\rangle$, $F|_{\mathrm{co}N}$ is closed in $\mathrm{co}N \times Y$, where co denotes the convex hull in the usual sense.

A set $K \subset Y$ is said to be σ- compact if K is a countable union of compact sets. A nonempty topological space is *acyclic* if all of its reduced Cech homology groups over rationals vanish.

A topological space Y is an *AR* (normal) if for every normal space X and every closed set $A \subset X$, each continuous function $f : A \to Y$ has an extension $F : X \to Y$.

A topological space Y is an *ANR* (normal) if for every normal space X and every closed set $A \subset X$, each continuous function $f : A \to Y$ can be extended over a neighborhood $U \supset A$.

The following notions were introduced by Dugundji [18], A subset Z of a topological space Y is called $k{-}PC_Y$ (k*-proximally connected in* Y) for $k \geq 0$, if for each neighborhood U of Z in Y, there exists a neighborhood $V \subset U$ of Z in Y such that the morphism induced on homotopy groups $i_* : \pi_k(V) \to \pi_k(U)$ is trivial, where $i : V \to U$ is the inclusion. Z is called PC_Y^n if it is $k{-}PC_Y$ for all k, $0 \leq k \leq n$. Z is PC_Y^∞ if it is PC_Y^n for all n.

The PC_Y^n is a condition on the embedding of Z in Y rather than on the structure of Z itself. Let us mention that given an ANR Y and a subset Z of Y, (i) if Z is an R_δ; that is, Z is an intersection of a decreasing sequence of compact ARs, then Z is PC_Y^n; (ii) if $Z = \cap_{i=1}^\infty Z_i$ is a decreasing sequence of compact PC_Y^n spaces, then Z is also PC_Y^n; if Z has trivial shape in Y—that is, Z is contractible in each of its neighborhoods in Y—then Z is also PC_Y^n.

Let X be a topological space and A a compact subspace of X. A collection $\{A_j\}_{j\in J}$ of compact subspaces of X indexed by a nonempty directed set J is called an *upper approximating family* of A, written $A = \overline{\lim}_j A_j$, if $A = \bigcap_{i\in J} A_j$ and the collection $\{A_j\}_{j\in J}$ is decreasing.

Let X be a set, Y a topological space, and $F : X \multimap Y$ a map. A collection $\{F_j : X \multimap Y\}_{j\in J}$ of maps indexed by a nonempty directed set J is called an *upper approximating family* for F, written $F = \overline{\lim}_j F_j$, if for each $x \in X$, $F(x) = \overline{\lim}_j F_j(x)$. For details, see [9],

A map $F : X \multimap Y$ is *compact* provided $F(X)$ is contained in a compact subset of Y; and *locally compact* if for every point $x \in X$ there exists a neighborhood U of

x such that the restriction $F|_U : U \multimap Y$ is compact. For a map $F : X \multimap X$, F^n denotes the n-th iteration of F defined by $F^n(x) = F(F^{n-1}(x))$ for $x \in X$.

Let $F : X \multimap X$ be u.s.c., $x \in X$, $K \subset X$, and $A \subset X$. We say that A *attracts* x if for each neighborhood U of A there is an $n \in \mathbb{N}$ such that $F^n(x) \subset U$, and that A is an *attractor* for F if it attracts all points in X. F is of *compact attraction* if it has a compact attractor and is locally compact. For details, see [20].

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

3. Generalized convex spaces

There have appeared many generalizations of the concept of convex subsets of a topological vector space (t.v.s.). Especially, the convex spaces due to Lassonde [35], and the H-spaces due to Horvath [30, 1, 2, 3] were shown to be very useful in many fields in mathematics such as the KKM theory, fixed point theory, coincidence theory, and others.

Motivated by recent works of Park on convex spaces [39, 40, 41, 42, 49, 50, 51, 52, 53, 54, 55, 56, 58] and H-spaces [46, 47, 48, 57], in this section, we introduce the notion of generalized convex spaces or G-convex spaces which include many of topological spaces having generalized convexity structures. In fact, generalized convex structures due to Michael [38], Pasicki [59, 60, 61], Komiya [34], Lassonde [35], Horvath [29, 30], Bielawski [12], Joó [31], Park [45, 46, 47, 48, 54] and many others are examples of our new concept.

Let Δ_n denote the standard n-simplex; that is,

$$\Delta_n = \{u \in \mathbb{R}^{n+1} : u = \sum_{i=0}^{n} \lambda_i(u)e_i,\ \lambda_i(u) \geq 0,\ \sum_{i=0}^{n} \lambda_i(u) = 1\},$$

where e_i is the i-th unit vector in $\mathbb{R}^{n+1}$. For each $u = \sum_{i=0}^{n} \lambda_i(u)e_i$, in Δ_n, the $(n+1)$-tuple $(\lambda_0(u), \cdots, \lambda_n(u))$ is called the *barycentric coordinate* of $u \in \Delta_n$. For a set A, let $|A|$ denote the cardinality of A.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X and a nonempty map $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n+1$, there exist a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J denotes the face of Δ_n corresponding to $J \in \langle A\rangle$.

We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D\rangle$. For an $(X, D; \Gamma)$, a subset C of X is said to be *G-convex* if for each $A \in \langle D\rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For a nonempty subset S of X, the *G-convex hull* of S, G-co S, is defined by

$$G - \text{co } S = \bigcap\{Y : S \subset Y \subset X \text{ and } Y \text{ is } G-\text{convex}\}.$$

Note that Γ_A does not need to contain A for $A \in \langle D\rangle$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$. Now we list various examples of G- convex spaces and give their fundamental properties.

3.1. **A convex subset of a t.v.s.**

A convex subset X in a t.v.s. is a G-convex space $(X; \Gamma)$ by putting $\Gamma_A = \text{co } A$.

3.2. **Michael's convex structure** [38]

If X is any set, and $i \leq n$, then $\partial_i : X^n \to X^{n-1}$ is defined by

$$\partial_i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}).$$

A *convex structure* on a metric space X with metric ρ assigns each positive integer n to a subset M_n of X^n and a function $k_n : M_n \times \Delta_{n-1} \to X$ such that

(1) if $x \in M_1$, then $k_1(x, 1) = x$;
(2) if $x \in M_n$ $(n \geq 2)$ and $i \leq n$, then $\partial_i x \in M_{n-1}$ and, for any $t \in \Delta_{n-1}$ with $t_i = 0$, $k_n(x, t) = k_{n-1}(\partial_i x, \partial_i t)$;
(3) if $x \in M_n$ $(n \geq 2)$ with $x_i = x_{i+1}$ for some $i < n$, and if $t \in \Delta_{n-1}$, then $k_n(x, t) = k_{n-1}(\partial_i x, t^*)$, where $t^* = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n-1})$;
(4) if $x \in M_n$, then the function $t \to k_n(x, t)$, from Δ_{n-1} to X, is continuous; and
(5) for all $\varepsilon > 0$ there exists a neighborhood V_ε of the diagonal in $X \times X$ such that, for all n and all $x, y \in M_n$, $(x_i, y_i) \in V_\varepsilon$ for $i = 0, \ldots, n-1$ implies $\rho(k_n(x, t), k_n(y, t)) < \varepsilon$ for all $t \in \Delta_{n-1}$.

For a subset S of a space X with convex structure, if $S^n \subset M_n$ for all n, then the *convex hull* of S, denoted by conv(S), is

$$\{k_n(x, t) : x \in S^n, t \in \Delta_{n-1}, n = 1, 2, \ldots\}.$$

By putting $\Gamma_A = \text{conv}(A)$ for all $A \in \langle X \rangle$ satisfying $A^n \subset M_n$ for all $n \in \mathbb{N}$ (or $n \leq |A|$), a metric space X with convex structure becomes a G-convex space $(X; \Gamma)$.

3.3. **Pasicki's S-contractible space** [59, 60, 61]

A topological space X is *S-contractible* if there is a map $S : X \times I \times X \to X$ such that for any $x \in X$, $\{S(x, t, \cdot)\}$ is a homotopy joining the identity with a constant function $S(x, 1, y) = x$.

For any nonempty set $A \subset X$, let

$$\text{co}S\ A = \bigcap \{D \subset X : A \subset D \text{ and for any } x \in A \text{ and } t \in I, S(x, t, D) \subset D\}.$$

For $A = \emptyset$, let co$S$ $A = \emptyset$. If co$S$ $A = A$, then A is called *S-convex.*

3.4. **Komiya's convex space** [34]

Let X be an arbitrary set. A function $p : 2^X \to 2^X$ is a *convex hull operation* on X whenever $p(\emptyset) = \emptyset$, $p(\{x\}) = \{x\}$ for $x \in X$, $p(A) = \bigcup \{p(F) : F \in \langle A \rangle\}$ for $A \subset X$, and $p(p(A)) = p(A)$.

A *convex space* (X, p, Ψ) consists of a topological space X, a convex hull operation p on X, and $\Psi = \{\varphi_F : F \in \langle X \rangle\}$, where $|F| = n + 1$, and $\varphi_F : \Delta_n \to p(F)$ is a homeomorphism which is convex-hull preserving; that is, $\varphi_F(\Delta_m) = p(A)$, where $A \subset F$ and $|A| = m + 1$.

A convex space (X, p, Ψ) becomes a G-convex space $(X; \Gamma)$ by putting $\Gamma_A = p(A)$ for $A \in \langle X \rangle$.

3.5. **Lassonde's convex space** [35]

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hull will be called a *polytope*. For details, see Lassonde [35]. Any convex space X becomes a G-convex space $(X;\Gamma)$ by putting $\Gamma_A = \text{co } A$.

Every convex subset of an affine space is a convex space when supplied with the induced topology. In particular, every convex subset of a t.v.s. or of a vector space with the finite topology is a convex space. A convex space need not be a subset of a t.v.s. For example, a vector space with an uncountable Hamel basis and the finite (i.e., weak) topology is a convex space but not a t.v.s. (see [4]).

3.6. **Park's convex space** [45, 54]

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any $N \in \langle D\rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. A subset A of (X, D) is said to be D-*convex* if, for any $N \in \langle D\rangle$, $D \subset A$ implies co $D \subset A$. If $X = D$, then $X = (X, X)$ reduced to a convex space in the sense of Lassonde. Note that for a convex space (X, D), X itself is not necessarily convex. For example, let X be any space containing an n-simplex Δ_n as a subspace and D the set of vertices of Δ_n. Then (X, D) is a convex space, and X may not be convex, but D- convex.

3.7. **Horvath's pseudoconvex space** [29]

Let X be a topological space and there is a function $S : X \times I \times X \to X$ such that

(1) $S(x, 0, y) = y$, $S(x, 1, y) = x$ for any $x, y \in X$.

A subset C of X is called S-*convex* if for all $(x, y, t) \in C \times I \times C$, $S(x, y, t) \in C$. Let $C_S\{A\} = \bigcap\{Y : A \subset Y \subset X \text{ and } Y \text{ is } S\text{-convex}\}$.

A pair (X, S) is called a *pseudoconvex space* by Horvath [29] if a function S satisfies (1) and

(2) for each $A \in \langle X\rangle$, $S|_{C_S(A)\times I\times C_S(A)}$ is continuous.

Note that Pasicki's S-contractible spaces and Horvath's pseudoconvex spaces are almost same.

3.8. **Bielawski's simplicial convexity** [12]

Let X be an arbitrary set. A family $\mathcal{C}$ of a subsets of X is called a *convexity* on X if $X \in \mathcal{C}$ and $\{A_i\}_{i\in J} \subset \mathcal{C}$ implies $\bigcap_{i\in J} A_i \in \mathcal{C}$. A function $p : 2^X \to 2^X$ is called a *convex prehull* on X if $A \subset p(A)$ for any $A \subset X$ and $A \subset B$ implies $p(A) \subset p(B)$ for any $A, B \subset X$. A convex pre-hull is called a *convex hull* if $p(p(A)) = p(A)$ for each $A \subset X$.

Let X be a topological space. Let us assume that for each $\{x_0, \cdots, x_n) \in X^{n+1}$, a continuous function $\Psi[x_0, \cdots, x_n] : \Delta_n \to X$ is given, and that the family $\Psi = \{\Psi[x_0, \cdots, x_n] : (x_0, \cdots, x_n) \in X^{n+1}, n \in \mathbb{N}\}$ satisfies the following:

(1) for all $x \in X$, $\Psi[x](1) = x$; and

(2) for all $n \geq 1$, $(x_0, \cdots, x_n) \in X^{n+1}$, $u \in \Delta_n$, and $i = 0, \cdots, n$,

$$u_i = 0 \Longrightarrow \{\Psi[x_0, \cdots, x_n](u) = \Psi[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n](\partial_i u),$$

where $\partial_i u = (u_0, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n)$.

The *simplicial convexity* $\mathcal{C}(\Psi)$ of a topological space X determined by the family Ψ is the convexity determined by the prehull $p_\Psi : 2^X \to 2^X$ defined as follows:

$$p_\Psi(A) = \{\Psi[a_0, \cdots, a_n](u) : n \in \mathbb{N}, (a_0, \cdots, a_n) \in A^{n+1}, u \in \Delta_n\}$$

for $A \subset X$, and

$$\mathcal{C}(\Psi) = \{A \subset X : p_\Psi(A) = A\}.$$

Remarks. 1. Note that, in the simplicial convexity,

$$p_\Psi(\{x\}) = \{\Psi[x](1)\} = \{x\}$$

for all $x \in X$.

2. A topological space X having Bielawski's simplicial convexity is a G-convex space $(X; \Gamma)$ by putting $\Gamma_A = p_\Psi(A)$ for all $A \in \langle X \rangle$.

3. Bielawski [12, Proposition (0,4)] noted that, if $\mathcal{C}$ is a convexity on X, then the function $\mathrm{conv}_{\mathcal{C}} : 2^{\mathrm{X}} \to 2^{\mathrm{X}}$ defined by

$$\mathrm{conv}_{\mathcal{C}}\mathrm{A} = \cap\{\mathrm{B} \in \mathcal{C} : \mathrm{A} \subset \mathrm{B}\}$$

is a convex hull on X and the convexity $\mathcal{C}_h = \{A \subset X : \mathrm{conv}_{\mathcal{C}}\mathrm{A} = \mathrm{A}\}$ is equal to $\mathcal{C}$.

Note that, in a G-convex space X, Γ_A may not be equal to G-co A for $A \in \langle X \rangle$. But G-co is a convex hull on X in the sense of Bielawski and $\mathcal{C}_G = \{A \subset X : G - \mathrm{co}A = A\} = \{A \subset X : A \text{ is } G\text{- convex}\}$.

3.9. **Horvath's c-structure** [30] **or H-spaces** [1, 2, 3]

A topoological space X is said to be *contractible* if the identity map 1_X of X is homotopic to a constant map.

Let X be a topological space. A *c-structure* on X is given by a map $F : \langle X \rangle \multimap X$ such that

(1) for all $A \in \langle X \rangle$, $F(A)$ is not empty and contractible; and

(2) for all $A, B \in \langle X \rangle$, $A \subset B$ implies $F(A) \subset F(B)$.

A pair (X, F) is then called a *c-space* by Horvath [30] and an *H-space* by Bardaro and Ceppitelli [1, 2, 3], For an H-space (X, F), a subset C of X is said to be *H-convex* (or *F-set*) if for each $A \in \langle C \rangle$, we have $F(A) \subset C$. Given a nonempty subset A of an H-space (X, F), the *H-convex hull* of A, denoted by H-co A, is defined by Tarafdar [64] as follows:

$$H\text{-co}A = \bigcap\{Y : A \subset Y \subset X \text{ and } Y \text{ is } H\text{-convex}\},$$

An H-space (X, F) is a G-convex space $(X; \Gamma)$. In fact, by putting $\Gamma_A = F(A)$ for each $A \in \langle X \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to X$ such that for all $J \subset A$, $\phi(\Delta_J) \subset F(J)$ by Horvath [30, Proposition (0.4)].

From now on, we write $(X; \Gamma)$ for H-spaces instead of (X, F) as in [1, 2, 3], where $\Gamma_A = F(A)$ for each $A \in \langle X \rangle$. For $A \in \langle X \rangle$, Γ_A does not need to contain A.

The following are examples of H-spaces hence of G-convex spaces.

(1) By putting $\Gamma_A =$co$S$ A for $A \in \langle X \rangle$, an S-contractible space is an H-space, as follows:

Lemma 3.1. *In an S-contractible space X, coS A is contractible for $A \subset X$.*

Proof. For $A \subset X$, let

$$\Delta = \{D \subset X : A \subset D, S(A, I, D) \subset D\}.$$

Then $S(A, I, \bigcap\{D : D \in \Delta\}) \subset D$ for each $D \in \Delta$. Hence $S(A, I, \bigcap\{D : D \in \Delta\}) \subset \bigcap\{D : D \in \Delta\}=\mathrm{co}S\ A$; that is, $S(x, I, \mathrm{co}S\ A) \subset \mathrm{co}S\ A$ for all $x \in A$; that is, $S(x, \cdot, \cdot) : I \times \mathrm{co}S\ A \to \mathrm{co}S\ A$ is well-defined. So $\mathrm{co}S\ A$ is contractible. □

Similarly, a pseudoconvex space X in the sense of Horvath is an H-space $(X;\Gamma)$ by putting $\Gamma_A = C_S\{A\}$ for $A \in \langle X\rangle$, since $C_S\{A\}$ is contractible.

(2) A convex space (X, p, Ψ) in the sense of Komiya is an H-space, since $\Gamma_A = p(A)$ is a homeomorphic image of Δ_n, where $|A| = n + 1$ and $A \in \langle X\rangle$.

(3)Any contractible space is an H-space:

(a) At first, we may put $\Gamma_A = X$ for $A \in \langle X\rangle$. With this structure, the only H-convex subset of X is X itself [1].

(b) A homotopy S between the projections $X \times I \times X \to X$ defines a pseudoconvex structure in the sense of Horvath and hence a c-structure. In fact, for any $A \in \langle X\rangle$, we may put $\Gamma_A = C_S\{A\}$.

(4) Let C be a convex set in a t.v.s., X a Hausdorff space, and $f : C \to X$ a continuous bijection. If $A \in \langle X\rangle$, then co $f^-(A)$ is a compact subset of C; therefore $f :$ co $f^-(A) \to f($ co $f^-(A))$ is an homeomorphism. Let $\Gamma_A = f($ co $f^-(A))$. Note that X could be a S^1, the torus, the Mőbius band, or the Klein bottle.

(5) Let $(E, || \cdot ||)$ be an infinite dimensional normed space and $\rho : E \to \mathbb{R}_+$, a positive function. Let $\Gamma_A = \overline{B}(0, \max_{a\in A}\rho(a))\backslash B(0, \min_{a\in A}\rho(a))$ for $A \in \langle E\rangle$, where $\overline{B}(0, r) = \{x \in E : ||x|| \le r\}$ and $B(0, r) = \{x \in E : ||x|| < r\}$. Then Γ_A is an absolute retract and is therefore contractible. See Dugundji and Granas [19].

(6) Let V be a vector space, X a subset of V, C a convex set contained in X, and $r : X \to C$ a function. For $A \in \langle X\rangle$, define $\Gamma_A =$ co $\{r(a) : a \in A\}$. Then Γ_A is contractible for any topology on X inducing the finite topology on Γ_A. For an arbitrary function $r : X \to C$, Γ may not be a closure operator.

(7) Let $(X, \le)$ be a topological space with a lattice structure such that for all $x_1, x_2 \in X$, the order interval $[x_1, x_2]$ is empty or contractible.

If $A = \{a_0, \cdots, a_n\}$, let $\Gamma_A = [a_0 \wedge \cdots \wedge a_n, a_0 \vee \cdots \vee a_n]$.

(8) A metric space (X, d) is *strongly convex* provided that for any two points x_1, x_2 of X, there is only one point $x_0 \in X$ such that $d(x_0, x_1) = d(x_0, x_2) = \frac{1}{2}d(x_1, x_2)$.

It can be shown that a compact connected strongly convex metric space has a natural c-structure given by a continuous function $\alpha : [0, 1] \times X \times X \to X$ such that $d(x_1, \alpha(t, x_1, x_2)) = (1 - t)d(x_1, x_2)$ and $d(x_2, \alpha(t, x_1, x_2)) = td(x_1, x_2))$.

(9) Let E be a Banach space, μ a non-atomic probability measure on a measurable space X, and $Y = L^1(X;E)$ the Banach space of μ- integrable functions $g : X \to E$. If $\{g_0, \cdots, g_n\} \subset Y$, let $\Gamma_{\{g_0,\cdots,g_n\}} = \{\Sigma_{i=0}^n \chi_{A_i} g_i : \{A_i\}_{i=0,\cdots,n}$ is a partition of X into μ-measurable sets$\}$. If Y is separable, then $\Gamma_{\{g_0,\cdots,g_n\}}$ is a retract of Y [13], and is therefore contractible.

Examples (4)-(9) are due to Horvath [30].

3.10. **Park's H-space** [46, 47, 48]

A triple $(X, D; \Gamma)$ is called an *H-space* if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $X = D$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$ which becomes an H-space in the sense of Horvath.

As Park remarked in [46], any homeomorphic image of a convex space is an H-space and hence a G-convex space.

3.11. **Joó's pseudoconvexity** [31]

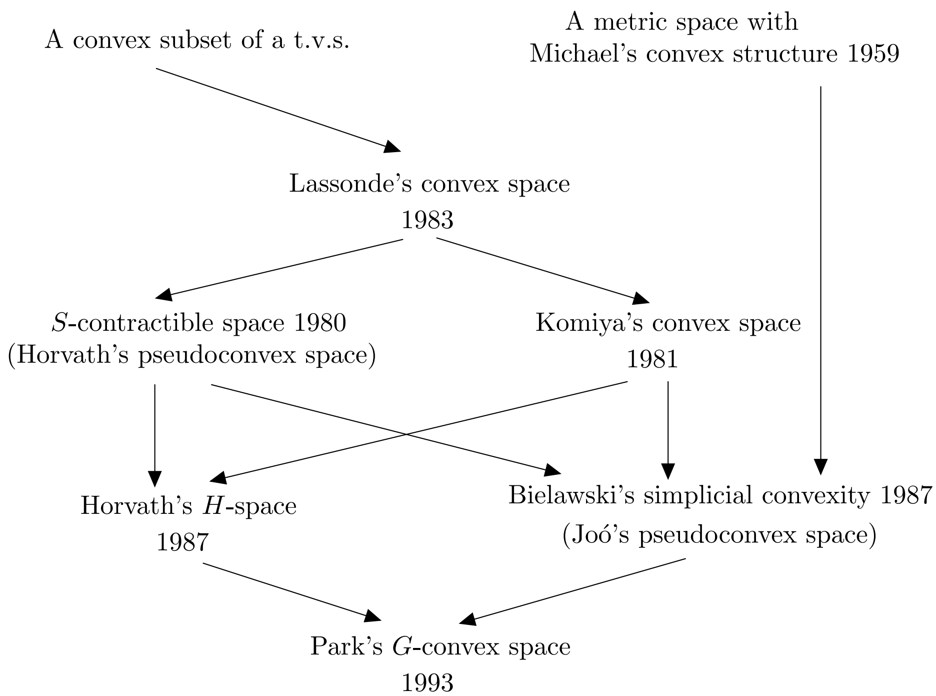
A *pseudoconvex space* (X, p, Ψ) consisits of a topological space X, a convex hull operation p on X defined as for Komiya's convex space, and $\Psi = \{\varphi_F : F \in \langle X \rangle\}$, where $\varphi_F : \Delta_n \to p(F)$ is a surjective continuous map and $|F| = n+1$ such that φ_F is convex-hull preserving: that is, for each face Δ_J of Δ_n corresponding $J \in \langle F \rangle$, we have

$$\varphi_F(\Delta_J) = p(J).$$

A topological space X with Joó's pseudoconvexity becomes a G-convex space $(X; \Gamma)$ by putting $\Gamma_A = p(A)$ for all $A \in \langle X \rangle$.

Joó and Stachó [32] defined the following convexity on $\mathbb{R}^{n+1}$: Let $x = (x_0, \cdots, x_n)$ and $y = (y_0, \cdots, y_n) \in \mathbb{R}^{n+1}$. We shall give the interval $\langle x, y \rangle$ joining them as a polygon with at most $n+1$ pairwise orthogonal segments as follows: If $x_n > y_n$, then let $I_n = \{(x_0, \cdots, x_{n-1}, t) : x_n \geq t \geq y_n\}$ and let $x' = (x_0, \cdots, x_{n-1}, y_n)$ be the other endpoint of I_n. If $x_n \leq y_n$, then let $I_n = \{(y_0, \cdots, y_{n-1}, t) : x_n \leq t \leq y_n\}$ and $y' = (y_0, \cdots, y_{n-1}, x_n)$. In the first case we get I_{n-1} analogously to I_n; if, for example $x_{n-1} \leq y_{n-1}$, then $I_{n-1} = \{(y_0, \cdots, y_{n-2}, t, y_n) : x_{n-1} \leq t \leq y_{n-1}\}$ and $y'' = (y_0, \cdots, y_{n-2}, x_{n-1}, y_n)$, if $x_{n-1} > y_{n-1}$, then $I_{n-1} = \{(x_0, \cdots, x_{n-2}, t, y_n) : x_{n-1} \geq t \geq y_{n-1}\}$ and $x'' = (x_0, \cdots, x_{n-2}, y_{n-1}, y_n)$. In the second case $(x_n \leq y_n)$ we construct analogously I_{n-1} and in the third step I_{n-2}, etc. Finally, the segments $I_0, \cdots, I_n$, parallel to the axis $x_0, \cdots, x_n$, resp., will join x and y (possibly not in the order of the indices). Now let a set $K \subset \mathbb{R}^{n+1}$ be *convex* if $x, y \in K$ implies $\langle x, y \rangle \subset K$. The space $\mathbb{R}^{n+1}$ with the convexity above is a pseudoconvex space and hence a G-convex space, see Joó [31].

The major particular forms of G-convex spaces can be adequately summarized by the following diagram. In the diagram, we may regard Horvath's pseudoconvex spaces as S-contractible spaces and Joó's pseudoconvex spaces as spaces with simplicial convexity, resp., for simplicity.



In the sequel, a convex space always means in the sense of Lassonde [35] or Park [45, 54], and an H-space always means in the sense of Horvath [30, 1, 2, 3] or Park [46, 47, 48],

4. Admissible classes of multimaps

Recently there have appeared many general classes of maps in the fixed point theory in nonlinear analysis or algebraic topology. In the long history of generalizations of the Brouwer fixed point theorem, Kakutani [33] first considered u.s.c. maps with nonempty compact convex values. This concept is extended by Eilenberg and Montgomery [21] to u.s.c. maps with compact acyclic values. Later Browder [14, 15, 16], Calvert [17], Górniewicz [23], Granas [26, 27], Ben-El-Mechaiekh et al. [5, 6, 7, 8, 9, 10, 11], Lassonde [36, 37], Dzedzej [20] and Park [50, 54, 55, 56] considered various classes of multimaps.

In 1992, Park initiated to unify major classes of maps which appeared in the above mentioned works. Actually, in a sequence of papers [45, 50, 54, 55, 56, 57], Park introduced "admissible" classes of maps denoted $\mathfrak{A}, \mathfrak{A}_c, \mathfrak{A}_c^\pi, \mathfrak{A}_c^\sigma, \mathfrak{A}_c^\kappa$, and constructed the fixed point theory and the KKM theory for those classes of maps.

The purpose in this section is to introduce such classes of maps and to study their basic properties, which will be used in our forthcoming works.

In this section, let X and Y be topological spaces and we may assume that every topological space is Hausdorff and that every map has nonempty values.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

A class $\mathfrak{A}_c^\kappa$ is defined as follows:

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any paracompact subset K of X, there exists an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}(x) \subset F(x)$ for all $x \in K$.

A class $\mathfrak{A}_c^\sigma$ is defined as follows:

$F \in \mathfrak{A}_c^\sigma(X,Y) \iff$ for any σ-compact subset K of X, there exists an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

$\mathfrak{A}_c^\sigma$ is due to Park [50, 54].

By the following due to Lassonde [37], $\mathfrak{A}_c^\sigma$ is closed under composition.

Proposition 4.1. *The image of a σ-compact set under a compact-valued u.s.c. map is σ-compact.*

Further, a class $\mathfrak{A}_c^\kappa$ is defined as follows:

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any compact subset K of X, there exists an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

$\mathfrak{A}_c^\kappa$ is due to Park [50, 54, 55, 56] and will be called *admissible.*

Proposition 4.2. $\mathfrak{A}_c^\kappa$ *is closed under composition.*

Now, we list various examples of admissible classes of maps and discuss their fundamental properties.

4.1. Examples of the class $\mathfrak{A}$

1. $f \in \mathbb{C}(X,Y) \iff f$ is continuous function from X into Y.

Note that $\mathbb{C} = \mathbb{C}_c$ and, for each polytope P, each $f \in \mathbb{C}_c(P,P)$ has a fixed point by the Brouwer fixed point theorem.

2. $F \in \mathbb{K}(X,Y) \iff F$ is a *Kakutani* map; that is, Y is a convex space and F is u.s.c. with compact convex values.

The class $\mathbb{K}$ is introduced by Kakutani [33].

Note that, for any polytope P, each $F = F_1 \cdots F_n \in \mathbb{K}_c(P,P)$ has a fixed point, where the range of F_i is contained in a convex space. This was noted by Simons [63], Lassonde [36], and Ben-El-Mechaiekh [5].

Hence $\mathbb{K}(X,Y)$ is an example of $\mathfrak{A}$ whenever Y is a convex space.

3. $F \in \mathbb{O}(X,Y) \iff F$ is u.s.c. with compact contractible values.

Note that $\mathbb{O}$ contains $\mathbb{K}$ if Y is a convex space.

4. $F \in \mathbb{O}_\delta(X,Y) \iff F = \overline{\lim}_{j\in J} F_j$ for some $F_j \in \mathbb{O}(X,Y)$.

5. $F \in \mathbb{R}_\delta(X,Y) \iff F$ is u.s.c. with R_δ-values.

6. $F \in \mathbb{D}(X,Y) \iff F$ is a *Dugundji* map; that is F is u.s.c. with compact PC_Y^∞ values.

Note that, if Y is an ANR space, then $\mathbb{D}(X,Y)$ contains the classes $\mathbb{O}(X,Y)$, $\mathbb{O}_\delta(X,Y)$, and $\mathbb{R}_\delta(X,Y)$ [9]. And $\mathbb{O}$, $\mathbb{O}_\delta$, $\mathbb{R}_\delta$, and $\mathbb{D}$ contain $\mathbb{C}$.

7. $F \in \mathbb{A}(X,Y) \iff F$ is an *approachable* map; that is, F is u.s.c. with compact values such that

(i) X and Y are subsets of t.v.s. E_1 and E_2, resp.; and

(ii) for any $U \in \mathcal{V}_{E_1}(0)$ and any $V \in \mathcal{V}_{E_2}(0)$, there is an $s \in \mathbb{C}(X,Y)$ such that

$$s(x) \in (F[(x+U) \cap X] + V) \cap Y \text{ for all } x \in X,$$

where $\mathcal{V}_{E_i}(0)$ denotes the neighborhood base of 0 in E_i.

Note that, for any polytope P, each $F \in \mathbb{A}_c(P, P)$ has a fixed point, where the intermediate spaces being arbitrary t.v.s. This is due to Ben-El-Mechaiekh and Deguire [9, Corollary (7.6)].

Moreover, since $\mathbb{D}(P, Y) \subset \mathbb{A}(P, Y)$ for a polytope P and a subset Y of a t.v.s. [9, Theorem 4.4], $\mathbb{O}$, $\mathbb{O}_\delta$, $\mathbb{R}_\delta$, $\mathbb{D}$, and $\mathbb{A}$ are examples of $\mathfrak{A}$.

There are more examples in [9] of classes belonging to $\mathbb{A}$.

8. $F \in \mathbb{V}(X, Y) \iff F$ is an *acyclic* map; that is F is u.s.c. with compact acyclic values.

The class $\mathbb{V}$ is introduced by Eilenberg and Montgomery [21].

For any polytope P, each $F \in \mathbb{V}_c(P, P)$ has a fixed point. This was noted by Powers [62], Górniewicz and Granas [24, 25], and Park [43].

Note that $\mathbb{C}(X, Y) \subset \mathbb{K}(X, Y) \subset \mathbb{O}(X, Y) \subset \mathbb{V}(X, Y)$ if Y is a convex space.

9. $F \in \mathbb{G}(X, Y) \iff F$ is a compact-valued u.s.c. map *admissible* in the sense of Calvert [17] and Górniewicz [23]; that is, there exists an $\tilde{F} \in \mathbb{V}_c(X, Y)$ such that $\tilde{F}(x) \subset F(x)$ for all $x \in X$.

Note that $\mathbb{V}(X, Y) \subset \mathbb{G}(X, Y)$.

10. $F \in \mathbb{P}(X, Y) \iff F$ is a compact-valued u.s.c. map *permissible* in the sense of Dzedzej [20]; that is, there is an $\tilde{F} \in M_c(K, Y)$ such that $\tilde{F}(x) \subset F(x)$ for all $x \in X$, where $M(X, Y)$ is given as follows:

Let $M_0(X, Y) = \mathbb{V}(X, Y)$;

$F \in M_m(X, Y) \iff F$ is continuous and has values consisting of one or m compact acyclic components; and

$F \in M(X, Y) \iff F \in M_m$ for some integer $m \geq 0$.

Note that $\mathbb{G}(X, Y) \subset \mathbb{P}(X, Y)$.

It is clear that $\mathbb{P}$ contains $\mathbb{C}$. Moreover, for any polytope P, $F \in \mathbb{P}_c(P, P)$ has a fixed point. This is a consequence of the following generalization of the Lefschetz fixed point theorem due to Dzedzej [20, Theorem 8.11]:

Theorem 4.3. *Let X be a metric ANR and $F \in \mathbb{P}(X, Y)$ a permissible map of compact attraction. Then the Lefschetz set $L(F)$ can be defined and $L(F) \neq \{0\}$ implies the existence of a fixed point of F.*

Therefore $\mathbb{P}$ and $\mathbb{G}$ are examples of $\mathfrak{A}$.

11. $F \in \mathbb{K}^*(X, Y) \iff Y$ is a convex space, F has compact convex values, and F is closed.

$F \in \mathbb{V}^*(X, Y) \iff F$ has compact acyclic values and F is closed.

$F \in \mathbb{K}^*_\omega(X, Y) \iff Y$ is a convex space, F has convex values, and F is demi-closed.

$F \in \mathbb{V}^*_\omega(X, Y) \iff F$ has acyclic values and F is demi-closed.

For details, see Granas and Liu [28].

If Y is compact or F is compact, then $\mathbb{K}^*(X, Y)$, $\mathbb{K}^*_\omega(X, Y)$, $\mathbb{V}^*(X, Y)$, and $\mathbb{V}^*_\omega(X, Y)$ are examples of $\mathfrak{A}$, since

$$\mathbb{K}(X, Y) = \mathbb{K}^*(X, Y) = \mathbb{K}^*_\omega(X, Y),$$
$$\mathbb{V}(X, Y) = \mathbb{V}^*(X, Y) = \mathbb{V}^*_\omega(X, Y)$$

by the following:

Lemma 4.4. *Let F be a map from X to 2^Y with closed values.*

(i) *If F is u.s.c., then F is closed.*

(ii) *If F is closed and Y is compact, then F is u.s.c.*

4.2. **Examples of the class $\mathfrak{A}_c^\pi$**

1. All of the examples of $\mathfrak{A}_c(X,Y)$.

In Examples 2-5, Y is a G-convex space.

2. $F \in \mathbb{R}(X,Y) \Longleftrightarrow F(x)$ is G-convex for every $x \in X$ and $F^-(y)$ is open for each $y \in Y$.

This class of maps is originated by Browder [14].

3. $F \in \mathbb{F}(X,Y) \Longleftrightarrow F(x)$ is G-convex for every $x \in X$ and $\{\mathrm{Int}F^-(y)\}_{y\in Y}$ covers X.

4. $F \in \mathbb{T}(X,Y) \Longleftrightarrow F(x)$ is G-convex and $\bigcup_{V\in\mathcal{V}(x)}\bigcap_{x'\in V} F(x') \neq \emptyset$ for each $x \in X$, where $\mathcal{V}(x)$ is the neighborhood base of x in X.

This class of maps is originated by Lassonde [37].

5. $F \in \Phi(X,Y) \Longleftrightarrow F(x)$ is G-convex for each $x \in X$ and there is a selection $\tilde{F}: X \multimap Y$ such that $\tilde{F}^-(y)$ is open for each $y \in Y$.

This class of maps is originated by Ben-El-Mechaiekh et al. [11].

6. $F \in \mathbb{M}^\pi(X,Y) \Longleftrightarrow$ for any paracompact subset K of X, $F|_K$ has a continuous selection.

By the Brouwer fixed point theorem, $\mathbb{M}^\pi$ is an example of $\mathfrak{A}_c^\pi$.

Proposition 4.5. *If X is a topological space and Y a G-convex space, then $\mathbb{R}(X,Y) \subset \mathbb{F}(X,Y) = \mathbb{T}(X,Y) = \Phi(X,Y)$.*

Proof. As in the proof of Lassonde [37, Proposition 4.2], we can show that $\mathbb{R} \subset \mathbb{T} = \Phi$ since the intersection of G-convex sets is G-convex also.

For $F \in \mathbb{T}(X,Y)$, define $\tilde{F}: X \multimap Y$ by $\tilde{F}(x) = \bigcup_{V\in\mathcal{V}(x)}\bigcap_{x'\in V} F(x')$. If $A = \{x_1,\cdots,x_n\} \in \langle\tilde{F}(x)\rangle$, then for each $i = 1,\cdots,n$, $x_i \in \bigcap_{x'\in V} F(x')$ for some $V_i \in \mathcal{V}(x)$. Put $V = \bigcap_{i=1}^n V_i \in \mathcal{V}(x)$. Then $A \in \langle\bigcap_{x'\in V} F(x')\rangle$ and G-co$A$ $\subset \bigcap_{x'\in V} F(x') \subset \tilde{F}(x)$ since $\bigcap_{x'\in V} F(x')$ is G-convex. Hence $\tilde{F}(x)$ is G-convex. And for each $y \in Y, x \in \tilde{F}^-(y) \Rightarrow y \in \bigcap_{x'\in V}\tilde{F}(x')$ for some $V \in \mathcal{V}(x) \Rightarrow y \in \tilde{F}(x')$ for all $x' \in V \Rightarrow x \in V \subset \tilde{F}^-(y)$, hence $\tilde{F}^-(y)$ is open. So $\tilde{F} \in \mathbb{R}(X,Y)$, $\tilde{F}(x) \subset F(x)$ for all $x \in X$ and

$$X = \bigcup_{y\in Y}\tilde{F}^-(y) = \bigcup_{y\in Y}\mathrm{Int}\tilde{F}^-(y) \subset \bigcup_{y\in Y}\mathrm{Int}F^-(y).$$

Hence $F \in \mathbb{F}(X,Y)$.

For $F \in \mathbb{F}(X,Y)$, suppose that $\bigcup_{V\in\mathcal{V}(x)}\bigcap_{x'\in V} F(x') = \emptyset$ for some $x \in X$. Since $X \subset \bigcup_{y\in Y}\mathrm{Int}F^-(y), x \in \mathrm{Int}F^-(y)$ for some $y \in Y$ and there exists a $V \in \mathcal{V}(x)$ such that $V \subset \mathrm{Int}F^-(y)$. Then $y \in \bigcap_{x'\in V} F(x')$, which is a contradiction. Therefore $F \in \mathbb{T}(X,Y)$. This completes our proof □

Let X be a topological space and Y an H-space. By Horvath [30, Theorem 3.2], $\Phi(X,Y) \subset \mathbb{M}^\pi(X,Y)$. So $\mathbb{R}(X,Y), \mathbb{F}(X,Y), \mathbb{T}(X,Y)$, and $\Phi(X,Y)$ are examples of $\mathfrak{A}_c^\pi(X,Y)$.

7. For a convex space Y, we define $\mathbb{S}(X,Y)$ and $\mathbb{S}_w(X,Y)$ as follows:

$F \in \mathbb{S}(X,Y) \Longleftrightarrow F$ is l.s.c. with closed convex values.

$F \in \mathbb{S}_w(X,Y) \Longleftrightarrow$ there exist two subsets Z, C of X such that $\dim_X(Z) \leq 0$, C is denumerable, and

(i) for each $x \in X\backslash C$, $F(x)$ is closed in Y;

(ii) for each $x \in X\backslash Z$, $F(x)$ is convex; and

(iii) F is l.s.c.

It is known that $\mathbb{S}(X,Y) \subset \mathbb{S}_w(X,Y) \subset \mathbb{M}^\pi(X,Y)$ if Y is a Banach space and that $\mathbb{S}(X,Y) \subset \mathbb{M}^\pi(X,Y)$ if Y is a complete metric space with Michael's convex structure. See Michael [38] or Horvath [30].

4.3. Examples of the class $\mathfrak{A}_c^\sigma$

1. Every example of $\mathfrak{A}_c^\pi$.
2. $F \in \mathbb{K}_c^\sigma(X,Y) \iff Y$ is a convex space and, for any σ-compact subset K of X, there is an $F \in \mathbb{K}_c(X,Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

$\mathbb{K}_c^\sigma$ is the same as $\mathbb{K}_c^+$ due to Lassonde [37].

Proposition 4.6. *If X and Y are convex subsets of a t.v.s., then $\mathbb{K}_c^\sigma$ contains the following classes:*

(1) *$\mathbb{M}^\pi(X,Y)$, hence $\mathbb{C}(X,Y), \mathbb{R}(X,Y)$ and $\mathbb{F}(X,Y)$.*
(2) *All maps $F : X \multimap Y$ of the form*

$$G : X = X_0 \xrightarrow{G_0} X_1 \xrightarrow{G_1} X_2 \longrightarrow \cdots \xrightarrow{G_n} X_{n+1} = Y,$$

where X_i $(1 \le i \le n)$ is a convex subset of a t.v.s. and each G_i $(0 \le i \le n)$ belongs to $\mathbb{K}(X_i, X_{i+1})$ or $\mathbb{T}(X_i, X_{i+1})$.

The class (2) of Proposition 4.6 is due to Lassonde [37].

3. $\mathbb{V}_c^\sigma(X,Y) \iff$ for any σ-compact subset K of X, there is an $\tilde{F} \in \mathbb{V}_c(X,Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

$\mathbb{V}_c^\sigma$ is the same as $\mathbb{V}_c^+$ due to Park, Singh, and Watson [58], and contains the following classes:

(1) $\mathbb{K}_c^\sigma(X,Y)$ if Y is a convex space.
(2) $\mathbb{G}$.

For any example $\mathbb{X}$ of $\mathfrak{A}$, we have $\mathbb{X} \subset \mathbb{X}_c \subset \mathbb{X}_c^\pi \subset \mathbb{X}^\sigma$, which are examples of $\mathfrak{A}_c^\sigma$.

4.4. Examples of the class $\mathfrak{A}_c^\kappa$

1. Every example of $\mathfrak{A}_c^\kappa$.
2. $F \in \mathbb{M}(X,Y) \iff Y$ is a convex space and, for any compact subset K of X, there exist a polytope $P \subset Y$ and a selection $s \in \mathbb{C}(K,P)$ of $F|_K$.

Note that $\mathbb{M}$ contains Φ [11, Théorème 2.2] if Y is a convex space.

3. $F \in \mathbb{K}_c^\kappa(X,Y) \iff Y$ is a convex space and for any compact subset K of X, there exists an $\tilde{F} \in \mathbb{K}_c(K,Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

Note that $\mathbb{K}_c^\kappa$ contains $\mathbb{M}$.

For any example $\mathbb{X}$ of $\mathfrak{A}$, we have $\mathbb{X} \subset \mathbb{X}_c \subset \mathbb{X}_c^\pi \subset \mathbb{X}_c^\sigma \subset \mathbb{X}_c^\kappa$, all of which are examples of $\mathfrak{A}_c^\kappa$.

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[a]The National Academy of Sciences, Republic of Korea, Seoul 137–044; and
Department of Mathematical Sciences, Seoul National University,
Seoul 151–747, KOREA
shparkmath.snu.ac.kr; parkcha38daum.net
[b]Department of Mathematical Education, Sehan University,
YoungAm-Gun, Chunnam, 526–702, KOREA
hoonjoosehan.ac.kr