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COINCIDENCES OF COMPOSITES OF ADMISSIBLE U.S.C. MAPS AND APPLICATIONS

by

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ABSTRACT. A composite of compact acyclic maps from a convex subset of a locally convex Hausdorff topological vector space into itself has a fixed point. This also holds for a very large class of u.s.c. multifunctions. Generalizations of this new result and their applications to the KKM theory are obtained.

1. Introduction and preliminaries

In this paper, we obtain general coincidence theorems for a very large class of upper semicontinuous multifunctions. From these, we obtain a new fixed point theorem and an extended form of the classical Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem. Consequently, our new results extend many of known theorems in [B, BD2, BDG1,2, GL, H, Hi, L1,2, P1,2, S] and others.

A *multifunction* $F : X \to 2^Y$ is a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-1}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For topological spaces X and Y, F is *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, $F^{-1}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$ is closed in X; and *compact* if $F(X) = \bigcup\{Fx : x \in X\}$ is contained in a compact subset of Y. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets [L1]. Such convex hull is called a *polytope*. Let D be a subset of X and $\langle D\rangle$ the set of all nonempty finite subsets of D. A multifunction $G : D \to 2^X$ is called a *KKM map* if $\operatorname{co} N \subset G(N)$ for each $N \in \langle D\rangle$.

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The *KKM theory* is the study of KKM maps and their applications. For the literature, see [P2,3].

Given a class $\mathbf{X}$ of multifunctions. we denote

$$\mathbf{X}(X, Y) = \{T : X \to 2^Y \mid T \in \mathbf{X}\};$$

$$\mathbf{X}_c = \{T = T_m T_{m-1} \cdots T_1 \mid T_i \in \mathbf{X}\}.$$

Let X be a convex space and Y a topological space. We define

$T \in \Phi(X, Y) \iff T^{-1}y$ is convex for each $y \in Y$ and $\{\operatorname{Int} Tx\}_{x \in X}$ covers Y.

$T \in \mathbf{M}(X, Y) \iff T^{-1}|_K$ has a continuous selection $s : K \to X$ for every nonempty compact subset K of Y such that $s(K) \subset P$ for some polytope P of X.

For topological spaces X and Y, we define

$f \in \mathbb{C}(X, Y) \iff f$ is a (single-valued) continuous function.

$T \in \mathbb{K}(X, Y) \iff T$ is a *Kakutani map*; that is, Y is a convex space and T is u.s.c. with nonempty compact convex values.

$T \in \mathbb{V}(X, Y) \iff T$ is an *acyclic map*; that is, T is u.s.c. with compact acyclic values.

Motivated by [BD2], we introduce an abstract class of multifunctions as follows:

$T \in \mathfrak{A}(X, Y) \iff T$ is an *admissible map*; that is, (i) $\mathfrak{A} \supset \mathbb{C}$; (ii) each $F \in \mathfrak{A}_c$ is u.s.c. and nonempty compact-valued; and (iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Note that $\mathbb{C}$, $\mathbb{K}$, and $\mathbb{V}$ are admissible classes. See [S, P1]. Moreover, the class of approachable maps in topological vector spaces is admissible. See [BD2]. For other examples of admissible classes, see [P3].

2. Coincidences of compact composites of admissible maps

We begin with the following coincidence theorem:

Theorem 1. *Let X be a convex space, Y a topological space, $F \in \mathfrak{A}_c(X, Y)$, and $G \in \mathbf{M}_c(X, Y)$. If F is compact, then F and G have a coincidence point $x_0 \in X$; that is, $Fx_0 \cap Gx_0 \neq \emptyset$.*

Proof. Since F is compact, there exists a compact set K such that $F(X) \subset K \subset Y$. Since $G \in \mathbf{M}_c(X, Y) = \mathbf{M}(X, Y)$ by [B, Lemma 2.1], $G^{-1}|_K$ has a continuous selection $s : K \to P$, where P is a polytope in X. Then $(sF)|_P \in \mathfrak{A}_c(P, P)$ has a fixed point $x_0 \in P$. Since $x_0 \in (sF)x_0$, we have $Fx_0 \cap Gx_0 \supset Fx_0 \cap s^{-1}(x_0) \neq \emptyset$. This completes our proof.

Particular forms of Theorem 1 are appeared in Ben-El-Mechaiekh *et al.* [BDG1,2, B], Ha [H], and Granas and Liu [GL].

Theorem 2. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c(X, Y)$ compact, and $G \in \Phi(X, \overline{F(X)})$. Then F and G have a coincidence point $x_0 \in X$.*

Proof. We show that $G \in \mathbb{M}(X, \overline{F(X)})$. In fact, since $\{\operatorname{Int} Gx : x \in X\}$ covers any compact subset K of $\overline{F(X)}$, there exists an $N = \{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that $\{\operatorname{Int} Gx : x \in N\}$ covers K. Let $\{\lambda_i\}_{i=1}^n$ be the partition of unity subordinated to this cover and $P = \operatorname{co} N \subset X$. Define $f : K \to P$ by

$$fy = \sum_{i=1}^{n} \lambda_i(y)x_i = \sum_{i \in N_y} \lambda_i(y)x_i \quad \text{for } y \in K,$$

where

$$i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in \operatorname{Int} Gx_i.$$

Then $x_i \in G^{-1}y$ for $i \in N_y$. Clearly f is continuous and $fy \in \operatorname{co}\{x_i : i \in N_y\} \subset G^{-1}y$ since $G^{-1}y$ is convex for each $y \in \overline{F(X)}$. Therefore, Theorem 2 follows from Theorem 1.

From Theorem 2, we have the following:

Theorem 3. *Let X and C be nonempty convex subsets of a locally convex Hausdorff topological vector space E, and $F \in \mathfrak{A}_c(X, X + C)$ a compact multifunction. Suppose that one of the following conditions holds:*

(i) *X is closed and C is compact.*

(ii) *X is compact and C is closed.*

(iii) *$C = \{0\}$.*

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Proof. Let V be an open convex neighborhood of the origin 0 in E, and Y a compact set such that $F(X) \subset Y \subset X + C$. Define $G : X \to 2^Y$ by $Gx = (x + C + V) \cap Y$ for $x \in X$. Then each Gx is open in Y and $G^{-1}y = (y - C - V) \cap X$ is convex for each $y \in Y$. Moreover, since $Y \subset X + C$, for every $y \in Y$, there exists an $x \in X$ such that $y \in x + C + V$; it follows that $\{Gx : x \in X\}$ covers Y. This shows $G \in \Phi(X, \overline{F(X)})$. Therefore, by Theorem 2, there exist $x_V \in X$ and $y_V \in Y$ such that $y_V \in Fx_V \cap Gx_V$; that is, $y_V - x_V \in C + V$. In other words, we obtain the assertion:

$(*)$ for each neighborhood V of 0 in E,

$$(F - i)(X) \cap (C + V) \neq \emptyset,$$

where $i : X \to E$ is the inclusion. Now we consider Cases (i)-(iii).

Case (i). Since X is closed so is $(F-i)(X)$. Since C is compact and E is regular, $(*)$ implies $(F-i)(X)\cap C\neq\emptyset$; that is, there exists an $\hat{x}\in X$ such that $F\hat{x}\cap(\hat{x}+C)\neq\emptyset$.

Case (ii). Since $(F-i)(X)$ is compact and C is closed, the same conclusion follows as in Case (i).

Case (iii). For each neighborhood V of 0 in E, there exist $x_V, y_V\in X$ such that $y_V\in Fx_V$ and $y_V-x_V\in V$. Since $F(X)$ is relatively compact, we may assume that y_V converges to some $\hat{x}$. Then x_V also converges to $\hat{x}$. Since the graph of F is closed in $X\times\overline{F(X)}$, we have $\hat{x}\in F\hat{x}$.

This completes our proof.

Particular forms of Theorem 3 were due to Lassonde [L1], Park [P2], Ben-El-Mechaiekh and Deguire [BD2], Simons [S], Himmelberg [Hi], and many others. For the literature, see [P3].

3. Non-compact case and the KKM theorem

Theorem 2 can be extended to the non-compact case as follows:

Theorem 4. *Let X be a convex space, D a nonempty subset of X, Y a Hausdorff space, $S: D\to 2^Y$, $T: X\to 2^Y$ multifunctions, and $F\in\mathfrak{A}_c(X,Y)$. Suppose that*

(1) *for each $x\in D$, $Sx\subset Tx$ and Sx is compactly open;*
(2) *for each $y\in F(X)$, $T^{-1}y$ is convex;*
(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)}\cap K\subset S(D)$; and*
(4) *for each $N\in\langle D\rangle$, there exists a compact convex subset L_N of X containing N such that $F(L_N)\backslash K\subset S(L_N\cap D)$.*

Then F and T have a coincidence point.

Proof. Since $\overline{F(X)}\cap K$ is compact, by (1) and (3), there exists an $N\in\langle D\rangle$ such that $\overline{F(X)}\cap K\subset S(N)$. Let L_N be the set in (4). We claim that $F(L_N)\subset S(L_N\cap D)$. Note that

$$F(L_N)\cap K\subset\overline{F(X)}\cap K\subset S(N)\subset S(L_N\cap D).$$

On the other hand, $F(L_N)\backslash K\subset S(L_N\cap D)$ by (4). Hence, we have $F(L_N)\subset S(L_N\cap D)$.

Note that $F(L_N)$ is compact since it is the image of the compact set L_N under the composite F of compact-valued u.s.c. multifunctions. Now we apply Theorem 2 with $F|_{L_N}$, $T|_{L_N}$, L_N and $F(L_N)$ replacing F, G, X and Y, respectively. Note that $T|_{L_N}$ has convex fibers $T^{-1}y\cap L_N$ for each $y\in F(L_N)$ by (2), and $S(L_N\cap D)$ covers $F(L_N)$; that is, $\{\operatorname{Int} Tx : x\in L_N\cap D\}$ covers $\overline{F(L_N)}=F(L_N)$, where $\operatorname{Int} Tx$ denotes the interior of Tx in $F(L_N)$. Hence, all of the requirements are satisfied, and thus $F|_{L_N}$ and $T|_{L_N}$ have a coincidence point $x_0\in L_N$. This completes our proof.

Theorem 4 is equivalent to the following generalization of the KKM theorem:

Theorem 5. *Let X be a convex space, D a nonempty subset of X, Y a Hausdorff space, and $F \in \mathfrak{A}_c(X, Y)$. Let $G : D \to 2^Y$ be a multifunction such that*

(5) *for each $x \in D$, Gx is compactly closed in Y;*

(6) *for any $N \in \langle D \rangle$, $F(\mathrm{co}\, N) \subset G(N)$; and*

(7) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof of Theorem 5 using Theorem 4. Let $S : D \to 2^Y$ and $T : X \to 2^Y$ be defined by $Sx = Y \backslash Gx$ for $x \in D$ and $T^{-1}y = \mathrm{co}\, S^{-1}y$ for $y \in Y$. Then (1) and (2) hold by (5) and the definitions of S and T. Suppose that $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} = \emptyset$; that is, $\overline{F(X)} \cap K \subset S(D)$. Then (3) holds. Note that (4) and (7) are equivalent. Therefore, by Theorem 4, F and T have a coincidence point $x_0 \in X$. For $y \in Fx_0 \cap Tx_0$, we have $x_0 \in T^{-1}y = \mathrm{co}\, S^{-1}y$ and hence, there exists an $N \in \langle S^{-1}y \rangle \subset \langle D \rangle$ such that $x_0 \in \mathrm{co}\, N$. Since $y \in Fx_0 \subset F(\mathrm{co}\, N) \subset G(N) = Y \backslash \bigcap\{Sx : x \in N\}$ by (6), $y \notin Sx$ for some $x \in N$; that is, $x \notin S^{-1}y$ and $x \in N$, which is a contradiction. This completes our proof.

Proof of Theorem 4 using Theorem 5. Let $Gx = Y \backslash Sx$ for $x \in D$. Then (5) and (7) follow from (1) and (4), resp. Moreover, from (3), we have

$$\overline{F(X)} \cap K \subset S(D) = \bigcup_{x \in D} (Y \backslash Gx) = Y \backslash \bigcap_{x \in D} Gx,$$

contrary to the conclusion of Theorem 5. Therefore, F and G do not satisfy (6) and hence, there exist an $N \in \langle D \rangle$ and a $y \in F(\mathrm{co}\, N) \backslash G(N)$; that is, $y \notin Gx$ or $y \in Sx$ for all $x \in N$. This implies $y \in Tx$ or $x \in T^{-1}y$ for all $x \in N$. Since $y \in F(\mathrm{co}\, N) \subset F(X)$, $T^{-1}y$ is convex by (2). Therefore, $\mathrm{co}\, N \subset T^{-1}y$. Note that $y \in Fx_0$ for some $x_0 \in \mathrm{co}\, N \subset T^{-1}y$, and hence $y \in Fx_0 \cap Tx_0$. This completes our proof.

Theorems 4 and 5 include very large number of known results and have many applications in the fixed point theory and the KKM theory. For the details, see [P3].

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