SOME COINCIDENCE THEOREMS ON ACYCLIC MULTIFUNCTIONS AND APPLICATIONS TO KKM THEORY

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ABSTRACT

From a Lefschetz type fixed point theorem for composites of acyclic maps, we obtain a general Fan-Browder type coincidence theorem, which can be shown to be equivalent to a matching theorem and a KKM type theorem. From the main result, we deduce the Himmelberg type fixed point theorem for acyclic compact multifunctions, acyclic versions of general geometric properties of convex sets, abstract variational inequality theorems, new minimax theorems, and non-continuous versions of the Brouwer and Kakutani type fixed point theorems with very generous boundary conditions.

1. Introduction

For multifunctions $S, T : X \to 2^Y$, the coincidence problem is to find sufficient conditions for the existence of points $(x_0, y_0) \in X \times Y$ satisfying $y_0 \in Sx_0 \cap Tx_0$. Since the pioneering work of von Neumann [**67**] in 1935, geometric problems of this type have attracted broad attention and remarkable progress has been made both

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in generalizing the known results as well as in finding new applications in a variety of mathematical areas. For the typical literature, see Granas [**38**], Dugundji and Granas [**23**], Aubin [**3**], Aubin and Ekeland [**3**], Lassonde [**59**], Fan [**32**], Granas and Liu [**37**], and Zeidler [**103**].

One of the main features of this progress is the applications of various equivalent formulations of the classical Knaster-Kuratowski-Mazurkiewicz theorem (simply, KKM theorem) in [**55**]. Since such applications are so broad and rich, this research area can be adequately called the KKM theory.

In 1961, using his own generalization of the KKM theorem, Ky Fan [**25**] established an elementary but very basic "geometric" lemma for multifunctions. Later, Browder [**15**] obtained this result in more convenient form of a fixed point theorem by means of the Brouwer fixed point theorem [**13**] and the partition of unity argument. Since then there have appeared numerous generalizations of the Fan-Browder fixed point theorem and their applications in various fields as coincidence and fixed point theory, minimax theory, variational inequalities, nonlinear analysis, convex analysis, game theory, mathematical economics, and many others.

In the present paper, we obtain a new coincidence theorem for a Browder type multifunction and an acyclic multifunction from a Lefschetz type fixed point theorem due to Górniewicz and Granas [**35**]. Using our new result, we obtain generalized and refined versions of basic results in KKM theory and the Brouwer or Kakutani type fixed point theorems.

Among our new results are matching theorems, new KKM theorems, the Himmelberg type theorem [**45**] for acyclic compact multifunctions, acyclic versions of general geometric properties of convex sets, abstract variational inequality theorems, new minimax theorems, and non-continuous versions of the Brouwer and Kakutani type theorems with very general boundary conditions. Consequently, main results in nearly one hundred published papers related to KKM theory are extended, improved, unified, and proved by drastically simplified methods.

2. Preliminaries

We follow mainly Berge [**10**] and Lassonde [**59**].

Let X and Y be sets. *A multifunction* $S : X \to 2^Y$ is a function from X into the power set 2^Y of Y, that is, $Sx \subset Y$ for each $x \in X$. For $A \subset X$, let $S(A) = \bigcup\{Sx : x \in A\}$. For $y \in Y$, let $S^-y = \{x \in X : y \in Sx\}$. For any $B \subset Y$, the *lower inverse* and *upper inverse* of B under S is defined by

$$S^-(B) = \{x \in X : Sx \cap B \neq \emptyset\} \quad \text{and} \quad S^+(B) = \{x \in X : \emptyset \neq Sx \subset B\},$$

resp.

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. In this paper, a convex hull of a nonempty finite subset of X will be called a *polytope*.

Let co, cl, Bd, and Int denote the convex hull, closure, boundary, and interior, resp.

A nonempty subset L of a convex space X is called a *c-compact set* if for each finite subset $N \subset X$, there is a compact convex subset of X containing $L \cup N$.

A subset B of a topological space Y is said to be *compactly closed* [resp. *open*] in Y if for every compact set $K \subset Y$ the set $B \cap K$ is closed [resp. open] in K.

An extended real-valued function $f : X \to \overline{\mathbf{R}}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is open for each $r \in \overline{\mathbf{R}}$; if X is a convex set in a vector space, then f is *quasi-concave* [resp. *quasi-convex*] whenever $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is convex for each $r \in \overline{\mathbf{R}}$.

Let D be a subset of a convex set X and $\mathcal{F}(D)$ the set of all nonempty finite subsets of D. A multifunction $G : D \to 2^X$ is called *KKM* if $\operatorname{co} N \subset G(N)$ for each $N \in \mathcal{F}(D)$.

For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if $F^+(V)$ is open for each open set $V \subset Y$; and *lower semicontinuous* (l.s.c.) if $F^-(V)$ is open for each open set $V \subset Y$. F is said to be *compact* if $F(X)$ is contained in a compact subset of Y.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic. For a topological space Y, $ka(Y)$ denotes the set of all compact acyclic subsets of Y, and $ca(Y)$ all closed acyclic subsets. For a convex space Y, $kc(Y)$ denotes the set of all nonempty compact convex subsets of Y, and $cc(Y)$ all nonempty closed convex subsets. An u.s.c. multifunction $F : X \to ka(Y)$ is usually called an *acyclic map*.

A topological vector space will be abbreviated by a t.v.s.

Let E be a real Hausdorff t.v.s. and E^* the topological dual of E; that is, the vector space of all continuous linear functionals on E. Let Y be a topological space and $F : Y \to 2^E$. Then

(i) F is *upper demicontinuous* (u.d.c.) if for each $y_0 \in Y$ and each open half-space H in E containing Fy_0, there exists an open neighborhood V of y_0 in Y such that $F(V) \subset H$; and

(ii) F is *upper hemicontinuous* (u.h.c.) if for each $f \in E^*$ and each $a \in \mathbf{R}$, the set $\{y \in Y : \sup f(Fy) < a\}$ is open in Y; that is, the function $\sup fF : Y \to \mathbf{R} \cup \{+\infty\}$ is u.s.c.

Note that u.s.c. $\Longrightarrow$ u.d.c. $\Longrightarrow$ u.h.c. and that if F is compact-valued, then u.d.c. $\Longleftrightarrow$ u.h.c. If $F, G : Y \to 2^E$ are u.h.c., so is $F + G$.

For a subset X in a t.v.s. E and for $x \in E$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, resp., are defined as follows :

$$I_X(x) = x + \bigcup_{r>0} r(X - x) \quad \text{and} \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

Their closures are denoted by $\overline{I}_X(x)$ and $\overline{O}_X(x)$, resp.

A multifunction $F : X \to 2^E$ is said to be *weakly inward* [resp. *outward*] if $Fx \cap \overline{I}_X(x) \neq \emptyset$ [$Fx \cap \overline{O}_X(x) \neq \emptyset$] for each $x \in \operatorname{Bd} X \backslash Fx$.

For each $f \in E^*$ and nonempty subsets $U, V \subset E$, let us denote

$$d_f(U, V) = \inf\{|f(u - v)| : u \in U,\ v \in V\}.$$

From the Lefschetz type fixed point theorems for composites of acyclic maps (Górniewicz and Granas [**35**], Granas and Liu [**37**]), we have the following :

Lemma 1. *Let X be a compact topological space and Y a convex space. If $f : X \to Y$ is a continuous function such that $f(X)$ is contained in a polytope P and $S : Y \to ka(X)$ is an u.s.c. multifunction, then fS has a fixed point $y_0 \in P$; that is, $y_0 \in (fS)y_0$.*

In [**40**], Ha obtained a particular form of Lemma 1 for $S : Y \to cc(X)$, where X is a convex space, and some applications.

We also need the following version of Fan's generalization [**25**] of the KKM theorem (See Dugundji and Granas [**22**] and Lassonde [**59**]).

Lemma 2. *Let D be a nonempty subset of a convex space X and $G : D \to 2^X$ a KKM multifunction with compactly closed values. If Gx_0 is compact for at least one $x_0 \in D$, then $\bigcap\{Gx : x \in D\} \neq \emptyset$.*

3. Coincidence theorems

We begin with the following general coincidence theorem.

Theorem 1. *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $S : D \to 2^Y$, $T : X \to 2^Y$ multifunctions, $F : X \to ka(Y)$ an u.s.c. multifunction, and K a nonempty compact subset of Y. Suppose that*

(1.1) *for each $x \in D$, $Sx \subset Tx$ and Sx is compactly open ;*

(1.2) *for each $y \in F(X)$, $T^{-}y$ is convex ;*

(1.3) $\operatorname{cl} F(X) \cap K \subset S(D)$ *; and*

(1.4) *for each $N \in \mathcal{F}(D)$, there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \backslash F^{+}(K)$ implies $Fx \subset S(L_N \cap D)$.*

Then T and F have a coincidence point $x_0 \in X$; that is, $Tx_0 \cap Fx_0 \neq \emptyset$.

Proof. Since $\operatorname{cl} F(X) \cap K$ is compact and covered by compactly open sets Sx by (1.1) and (1.3), there exists a finite subset N of D such that $\operatorname{cl} F(X) \cap K \subset S(N)$. Consider the set L_N in (1.4).

We first claim that $F(L_N) \subset S(L_N \cap D)$. In fact, if $x \in L_N \cap F^{+}(K)$, then $Fx \subset K$ and

$$Fx \subset F(L_N) \cap K \subset F(X) \cap K \subset S(N) \subset S(L_N \cap D).$$

On the other hand, if $x \in L_N \backslash F^{+}(K)$, then $Fx \subset S(L_N \cap D)$ by (1.4). Therefore, $F(L_N) \subset S(L_N \cap D)$.

Note that $F(L_N)$ is compact since it is the image of the compact set L_N under the compact-valued u.s.c. multifunction F.

We show that $T^{-} \,|\, F(L_N) : F(L_N) \to 2^X$ has a continuous selection $f : F(L_N) \to C$ where $C = \operatorname{co}\{x_1, x_2, \cdots, x_n\} \subset L_N$ for some $x_1, x_2, \cdots, x_n \in L_N$. In fact, since $F(L_N)$ is compact and included in $S(L_N \cap D)$, by (1.1), we may assume that $F(L_N) \subset \bigcup_{i=1}^{n} Sx_i$ for $x_i \in L_N \cap D$. Let $\{\lambda_i\}_{i=1}^{n}$ be the partition of unity subordinate to this cover, and let $C = \operatorname{co}\{x_1, x_2, \cdots, x_n\} \subset X$. Define $f : F(L_N) \to C$ by

$$fy = \sum_{i=1}^{n} \lambda_i(y) x_i = \sum_{i \in N_y} \lambda_i(y) x_i$$

for $y \in F(L_N) \subset F(X)$, where

$$i \in N_y \Longleftrightarrow \lambda_i(y) \neq 0 \Longrightarrow y \in Sx_i.$$

Then $x_i \in S^{-}y \subset T^{-}y$ by (1.1). Clearly f is continuous and, since $T^{-}y$ is convex by (1.2), we have $fy \in \operatorname{co}\{x_i : i \in N_y\} \subset T^{-}y$ for each $y \in F(L_N)$.

Since $C \subset L_N$ and $F \,|\, C : C \to ka(F(L_N))$ is u.s.c., by Lemma 1, $f(F \,|\, C) : C \to 2^C$ has a fixed point $x_0 \in C \subset L_N$. Since $x_0 \in (fF)x_0$ and $f^{-1}x_0 \subset Tx_0$, we have $Tx_0 \cap Fx_0 \neq \emptyset$. This completes our proof.

Remarks. 1. Note that the Hausdorffness of Y is necessary for the partition of unity argument in the proof. Later in Section 5, we show that the Hausdorffness of Y is not necessary if F is single-valued. Note that (1.2) may be replaced by the convexity of $T^{-}y$ for each $y \in F(L_N)$.

2. Note that if F is compact (e.g., X is compact or Y is compact), then (1.4) holds automatically. As we can notice in the proof, the N in (1.4) is not necessarily arbitrary, but satisfies $\operatorname{cl} F(X) \cap K \subset S(N)$. Moreover, if there exists a c-compact subset L of X, then the L_N in (1.4) can be chosen the closed convex hull of $L \cup N$ in X. The coercivity condition (1.4) is motivated by Chang [**19**].

Particular forms. See Park [**71**, Theorem 1], [**70**, Theorem 6, where A should be $A : X \to 2^Y$ and $x_0 \in X$], [**74**, Theorem 7], and Chang [**19**], Theorems 2.4 and 2.7.

We will frequently use the following particular form of Theorem 1 with $D = X$.

Corollary 1.1. *Let X be a convex space, Y a Hausdorff space, $S, T : X \to 2^Y$ multifunctions, $F : X \to ka(Y)$ an u.s.c. multifunction, and K a nonempty compact subset of Y. Suppose that*

(1) *for each $x \in X$, Sx is compactly open ;*

(2) *for each $y \in F(X)$, $\operatorname{co} S^{-}y \subset T^{-}y$;*

(3) $\operatorname{cl} F(X) \cap K \subset S(X)$*; and*

(4) *for each $N \in \mathcal{F}(X)$, there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \backslash F^{+}(K)$ implies $Fx \subset S(L_N)$.*

Then there exists an $x_0 \in X$ such that $Tx_0 \cap Fx_0 \neq \emptyset$.

Proof. Note that, for $X = D$, (4) is equivalent to (1.4).

Particular forms. See Browder [**15**, Theorems 1 and 7], [**16**, Proposition 1], [**17**, Theorems 2 and 5], Tarafdar [**95**, Theorem 1], [**96**, Corollary 2.1 and Theorem 2.2], [**97**, Theorem 1.2], [**98**, Theorem 2], Tarafdar and Husain [**98**, Theorem 1.1], Ben-El-Mechaiekh, Deguire, and Granas [**8**, Théorème 1], [**9**, I, Théorème 1, 2, et 5; II, Théorèmes 3.1–3.3 et 4.1], Yannelis and Prabhakar [**101**, Theorems 3.2 and 3.3], Lassonde [**59**, Theoerm 1.1], Ko and Tan [**56**, Theorem 3.1], Simons [**88**, Theorem 4.3], Takahashi [**93**, Theorems 2 and 5], Komiya [**57**, Theorem 1], Mehta [**64**, Theorem 3.1], Mehta and Tarafdar [**65**, Theorems 1–5], Sessa [**82**, Theorems 4,7, and 8], Jiang [**48**, I, Lemma 3.2], [**49**, Lemma 2.1], [**50**, Corollary 3.2], McLinden [**63**, Theorem], and Park [**71**, Theorems 2 and 9].

If the multifunction F in Corollary 1.1 is compact, we have the following :

Corollary 1.2. (Granas and Liu [**37**, Theorem 4.1]) *Let X be a convex space, Y a Hausdorff space, and $T, F : X \to 2^Y$ multifunctions such that*

(1) *$T^{-}y$ is convex for each $y \in Y$;*

(2) *$\{\operatorname{Int} Tx\}_{x\in X}$ covers Y ; and*

(3) *F is compact and, for each polytope C in X, the restriction $F\,|\,C : C \to ka(Y)$ is u.s.c.*

Then T and F have a coincidence point.

Proof. Let K be a compact subset of Y such that $F(X) \subset K \subset Y$. Since $K \subset \bigcup_{i=1}^{n} \operatorname{Int} Tx_i$ for a finite subset $\{x_1, x_2, \cdots, x_n\}$ in X. Let $C = \operatorname{co}\{x_1, x_2, \cdots, x_n\}$ and $Sx = \operatorname{Int} Tx$ for each $x \in C$. Apply Corollary 1.1 with $(C, K, S, T\,|\,C, F\,|\,C)$ instead of (X, Y, S, T, F). Then all the requirements of Corollary 1.1 are clearly satisfied. Therefore, the conclusion follows.

Remarks. 1. Corollary 1.2 is the main result of Granas and Liu [**36**, Théorème 2.1], [**37**, Theorem 4.1], and most of other results in [**37**], especially, coincidence or fixed point theorems in Section 4 of [**37**], follow from Corollary 1.2.

2. In fact, Granas and Liu [**37**] assumed that X is a convex subset of a vector space with the finite topology. Then X becomes a convex space and, in their case, F is u.s.c. if and only if $F\,|\,C$ is u.s.c. for every polytope C in X. Therefore [**37**, Theorem 4.1] directly follows from Corollary 1.1.

4. Matching theorems and KKM theorems

In this section, we obtain far-reaching generalizations of the Ky Fan type matching theorems for open coverings and the KKM type theorems.

Theorem 2. *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $S : D \to 2^Y$ a multifunction, $F : X \to ka(Y)$ an u.s.c. multifunction, and K a nonempty compact subset of Y. Suppose that*

(2.1) *for each $x \in D$, Sx is compactly open ;*

(2.2) $\operatorname{cl} F(X) \cap K \subset S(D)$ *; and*

(2.3) *for each $N \in \mathcal{F}(D)$, there exists an $L_N \in kc(X)$ as in* (1.4).

Then there exist an $\{x_1, x_2, \cdots, x_n\} \in \mathcal{F}(D)$ and an $x_0 \in \operatorname{co}\{x_1, x_2, \cdots, x_n\}$ such that $Fx_0 \cap \bigcap_{i=1}^{n} Sx_i \neq \emptyset$.

Proof. For each $y \in Y$, put $T^{-}y = \operatorname{co} S^{-}y$. This defines a multifunction $T : X \to 2^Y$. Then all of the requirements of Theorem 1 are satisfied. In fact, for each $x \in D$, $x \in S^{-}y$ for some $y \in Y$ implies $x \in T^{-}y$, that is, $Sx \subset Tx$. This and (2.1) imply (1.1). Moreover, $T^{-}y$, which may be empty, is convex for each $y \in F(X)$. This shows (1.2). Since (2.2) and (2.3) are same as (1.3) and (1.4), resp., by Theorem 1, T and F have a coincidence point $x_0 \in X$, that is, $Tx_0 \cap Fx_0 \neq \emptyset$. For $y \in Tx_0 \cap Fx_0$, we have $x_0 \in T^{-}y = \operatorname{co} S^{-}y$, and hence, there exists a finite set $\{x_1, x_2, \cdots, x_n\}$ in $S^{-}y \subset D$ such that $x_0 \in \operatorname{co}\{x_1, x_2, \cdots, x_n\}$. Since $x_i \in S^{-}y$ implies $y \in Sx_i$ for all i, $1 \leq i \leq n$, we have $y \in Fx_0 \cap \bigcap_{i=1}^{n} Sx_i$. This completes our proof.

In fact, Theorems 1 and 2 are equivalent.

Proof of Theorem 1 using Theorem 2. Since (1.3) and (1.4) are same as (2.2) and (2.3), resp., and (1.1) implies (2.1), by Theorem 2, there exist $\{x_1, x_2, \cdots, x_n\} \subset D$ and $x_0 \in \operatorname{co}\{x_1, x_2, \cdots, x_n\}$ such that $Fx_0 \cap \bigcap_{i=1}^{n} Sx_i \neq \emptyset$. For $y \in Fx_0 \cap \bigcap_{i=1}^{n} Sx_i$, we have $y \in \bigcap_{i=1}^{n} Sx_i \subset \bigcap_{i=1}^{n} Tx_i$ by (1.1). Since $y \in Fx_0 \subset F(X)$ and $x_i \in T^{-}y$ for all i, $1 \leq i \leq n$, by (1.2), $\operatorname{co}\{x_1, x_2, \cdots, x_n\} \subset T^{-}y$. In particular, $x_0 \in T^{-}y$, that is, $y \in Tx_0$. Hence $y \in Tx_0 \cap Fx_0$. This completes our proof.

Particular forms. Theorem 2 generalizes Fan's matching theorem for open coverings in [**32**]. In fact, if F is single-valued, then Theorem 2 contains [**70**, Theorems 1 and 2] and [**74**, Theorem 2]. Further, if $X = Y$ and $F = 1_X$, the identity function, Theorem 2 includes Fan [**32**, Lemma 1 and Theorem 3].

Theorem 2 can be restated in its contrapositive form and in terms of the complement Gx of Sx in Y as follows :

Theorem 3. *Let D, X, Y, F, and K be as in Theorem 2. Let $G : D \to 2^Y$ be a multifunction such that*

(3.1) *for each $x \in D$, Gx is compactly closed ;*

(3.2) *for each $N \in \mathcal{F}(D)$, $F(\operatorname{co} N) \subset G(N)$; and*

(3.3) *there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \backslash F^{+}(K)$ implies $\bigcap\{Gz : z \in L_N \cap D\} \subset Y \backslash Fx$.*

Then $\operatorname{cl} F(X) \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof. Suppose the conclusion does not hold. Then $\operatorname{cl} F(X) \cap K \subset S(D)$, where $Sx = Y \backslash Gx$ for $x \in D$. Since $\operatorname{cl} F(X) \cap K$ is compact, there exists a finite subset $N \subset D$ such that $\operatorname{cl} F(X) \cap K \subset S(N)$. Therefore, by Theorem 2, there exists an $M \in \mathcal{F}(D)$ such that $F(\operatorname{co} M) \cap \bigcap\{Sx : x \in M\} \neq \emptyset$; that is, $F(\operatorname{co} M) \not\subset G(M)$. This contradicts (3.2).

Remark. For $X = Y$ and $F = 1_X$, (3.2) simply states that $G : D \to 2^X$ is a KKM multifunction. In fact, Theorem 3 generalizes a number of known KKM type theorems.

Particular forms. For a single-valued function $F = f : X \to Y$, Theorem 3 reduces to [**74**, Theorem 3], which includes [**70**, Theorems 3 and 4], [**74**, Theorem 4]. Moreover, Theorem 3 generalizes Chang [**19**, Theorem 2.1], Lassonde [**59**, Theorems I and III], and Fan [**25**, Lemma 1], [**30**, Theorem 1], [**32**, Theorem 4].

Remark. Actually, Lemma 2 follows from Theorem 3. In fact, put $X = Y$, $F = 1_X$, $K = Gx_0$, and $L_N = \mathrm{co}(\{x_0\} \cup N)$ in Theorem 3. Then (3.3) is satisfied since $x \notin Gx_0$ implies $\bigcap\{Gz : z \in L_N \cap D\} \subset Gx_0 \subset X \backslash \{x\}$. Therefore, $Gx_0 \cap \bigcap\{Gx : x \in D\} = \bigcap\{Gx : x \in D\} \neq \emptyset$.

However, according to Theorem 3, we have to assume the Hausdorffness of X in Lemma 2, which is superfluous. We discuss this matter in the next section.

5. Some general KKM theorems

Since Theorems 1–3 are mutually equivalent, we shall focus mainly on Theorem 3 in this section. We first need the following, which can be deduced from Lemma 2.

Lemma 3. (Jiang [**50**, Lemma 1]) *Let D be a nonempty subset of a convex space X and $G : D \to 2^X$ a KKM multifunction with compactly closed values. Then for every nonempty compact convex subset X_0 of X, we have*

$$X_0 \cap \bigcap\{Gx : x \in X_0 \cap D\} \neq \emptyset.$$

Proof. Define $G_0x = Gx \cap X_0$ for $x \in X_0 \cap D$. Then $G_0 : X_0 \cap D \to 2^{X_0}$ is well-defined. If $X_0 \cap D = \emptyset$, then the conclusion holds trivially. Therefore, we may assume $X_0 \cap D \neq \emptyset$. Consider $(X_0 \cap D,\, X_0,\, G_0)$ instead of (D, X, G) in Lemma 2. Then all of the requirements of Lemma 2 are satisfied. Therefore we have $\bigcap\{G_0x : x \in X_0 \cap D\} \neq \emptyset$ or $X_0 \cap \bigcap\{Gx : x \in X_0 \cap D\} \neq \emptyset$. This completes our proof.

Remark. Jiang obtained Lemma 3 from the Brouwer fixed point theorem and the partition of unity argument.

Theorem 4. *Let D be a nonempty subset of a convex space X, Y a topological space, $G : D \to 2^Y$, $F : X \to 2^Y$ multifunctions, and K a nonempty compact subset of Y. Suppose that*

(4.1) *for each $x \in D$, Gx and F^+Gx are compactly closed ;*

(4.2) *$F^+G : D \to 2^X$ is a KKM multifunction ;*

(4.3) *there exists a multifunction $H : D \to 2^X$ satisfying*

(a) *$Hx \subset F^+Gx$ for each $x \in D$;*

(b) *for each $N \in \mathcal{F}(D)$, there exists an $L_N \in kc(X)$ containing N such that $L_N \cap \bigcap\{Hx : x \in L_N \cap D\} \subset F^+K$; and*

(c) *$L_N \cap \bigcap\{Hx : x \in L_N \cap D\} = \emptyset$ implies $L_N \cap \bigcap\{F^+Gx : x \in L_N \cap D\} \subset F^+K$.*

Then we have $\operatorname{cl} F(X) \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof. We use Lemma 3. Suppose the conclusion does not hold. Since $\operatorname{cl} F(X) \cap K$ is compact, there exists an $N \in \mathcal{F}(D)$ such that

$$F(X) \cap K \subset \operatorname{cl} F(X) \cap K \subset \bigcup_{x \in N} (Y \backslash Gx).$$

Since there exists an $L_N \in kc(X)$, we have

$$L_N \cap \bigcap_{x \in L_N \cap D} F^+Gx \cap F^+(K) = \emptyset,$$

and hence

$$L_N \cap \bigcap_{x \in L_N \cap D} Hx \cap F^+(K) = \emptyset$$

by (a). Since

$$L_N \cap \bigcap_{x \in L_N \cap D} Hx \subset F^+(K)$$

by (b), we have

$$L_N \cap \bigcap_{x \in L_N \cap D} Hx = \emptyset,$$

which implies

$$L_N \cap \bigcap_{x \in L_N \cap D} F^+Gx \subset F^+(K)$$

by (c). Therefore, we must have

$$L_N \cap \bigcap_{x \in L_N \cap D} F^+Gx = \emptyset,$$

which contradicts Lemma 3. This completes our proof.

Remarks. 1. The condition (4.1) is satisfied if we assume one of the following :

(i) F is l.s.c. and Gx is closed for each $x \in D$.

(ii) F is a compact-valued continuous multifunction and Gx is compactly closed for each $x \in D$.

(iii) $F = t : X \to Y$ is a single-valued continuous function and Gx is compactly closed for each $x \in D$.

2. The condition (4.2) implies (3.2), but not conversely.

3. For the above case (iii), Theorem 4 reduces to Jiang [**50**, Theorem 2.2], which unifies three basic KKM type theorems of Lassonde [**59**, Theorems I, II, and III], and thus includes Dugundji and Granas [**22**, Corollary 1.4]. For details, see [**50**]. Note that our condition (b) is much weaker than Jiang's.

In case $H = F^+G$, Theorem 4 reduces to the following :

Theorem 4′. *Let D, X, Y, G, F, and K be as in Theorem 4. Suppose that*

(4.1) *for each $x \in D$, Gx and F^+Gx is compactly closed ;*

(4.2) *$F^+G : D \to 2^X$ is a KKM multifunction ; and*

(4.3)′ *for each $N \in \mathcal{F}(D)$, there exists an $L_N \in kc(X)$ containing N such that*

$$L_N \cap \bigcap\{F^+Gx : x \in L_N \cap D\} \subset F^+K.$$

Then we have $\operatorname{cl} F(X) \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Remark. As we noted above, Theorem 4′ cannot be comparable to Theorem 3. However, for the case (iii), (3.1)–(3.3) are same as (4.1), (4.2), and (4.3)′, resp. Only nontrivial part is to show (3.3) is same as (4.3)′. In fact, both conditions can be written as follows :

(3.3) $x \in L_N,\ Fx \cap \bigcap\{Gz : z \in L_N \cap D\} \neq \emptyset \Longrightarrow x \in F^+(K)$.

(4.3)′ $x \in L_N,\ Fx \subset \bigcap\{Gz : z \in L_N \cap D\} \neq \emptyset \Longrightarrow x \in F^+(K)$.

Therefore, in general, we have (3.3) $\Longrightarrow$ (4.3)′. However, for single-valued F, (3.3) is same as (4.3)′. Therefore, for single-valued F, Theorems 3 and 4′ are the same and the Hausdorffness assumption on Y in Theorem 3 is superfluous. Since Theorems 1–3 are equivalent, the same ramark also works for Theorems 1 and 2 and their corollaries, if any.

6. Analytic alternatives and fixed point theorems

Actually, Theorem 1 can be restated in various equivalent formulations. However, for the sake of simplicity, we consider the particular case Corollary 1.1 from now on. In this section, we give an equivalent form of Corollary 1.1 and two direct applications. The first one is a basis of various minimax inequalities and the second one generalizes the Himmelberg fixed point theorem [**45**, Theorem 2] to acyclic compact multifunctions.

Theorem 5. *Let X be a convex space, Y a Hausdorff space, $F : X \to ka(Y)$ an u.s.c. multifunction, $A, B \subset Z$ sets, $f, g : X \times Y \to Z$ functions, and K a nonempty compact subset of Y. Suppose that*

(5.1) *for each $x \in X$, $\{y \in Y : g(x, y) \in A\}$ is compactly open ;*

(5.2) *for each $y \in F(X)$, $\{x \in X : f(x, y) \in B\} \supset \operatorname{co}\{x \in X : g(x, y) \in A\}$; and*

(5.3) *for each $N \in \mathcal{F}(X)$, there exists an $L_N \in kc(X)$ containing N such that, for each $x \in L_N \backslash F^+(K)$ and each $y \in Fx$, there exists an $x_1 \in L_N$ satisfying $g(x_1, y) \in A$.*

Then either

(a) *there exists a $\hat{y} \in \operatorname{cl} F(X) \cap K$ such that $g(x, \hat{y}) \notin A$ for all $x \in X$, or*

(b) *there exists an $\hat{x} \in X$ and a $\hat{y} \in F\hat{x}$ such that $f(\hat{x}, \hat{y}) \in B$.*

Proof of Theorem 5 using Corollary 1.1. Consider the multifunctions $S, T : X \to 2^Y$ given by

$$Sx = \{y \in Y : g(x, y) \in A\} \quad \text{and} \quad Tx = \{y \in Y : f(x, y) \in B\}$$

for each $x \in X$. Then the requirements (1) and (2) of Corollary 1.1 are satisfied. Suppose that (a) does not hold. This means that $\operatorname{cl} F(X) \cap K$ is covered by Sx's. Since $\operatorname{cl} F(X) \cap K$ is compact, there exists an $N \in \mathcal{F}(X)$ such that $\operatorname{cl} F(X) \cap K \subset S(N)$, whence we have (3). Further, (5.3) implies the requirement (4). Therefore, by Corollary 1.1, there exists a point $\hat{x} \in X$ such that $T\hat{x} \cap F\hat{x} \neq \emptyset$; that is, there exists a $\hat{y} \in F\hat{x}$ such that $\hat{y} \in T\hat{x}$. This completes our proof.

Proof of Corollary 1.1 using Theorem 5. Assume the hypothesis of Corollary 1.1, and apply Theorem 5 to $Z = X \times Y$. Let A be the graph of S with $g(x, y) = (x, y)$, and B the graph of T with $f(x, y) = (x, y)$. Then (1), (2), and (4) in Corollary 1.1 imply (5.1)–(5.3) in Theorem 5. However, Property (a) in Theorem 5 does not hold since $S^- y$ is assumed nonempty for each $y \in \operatorname{cl} F(X) \cap K$ by (3) in Corollary 1.1. Hence Property (b) holds, that is, there exist an $\hat{x} \in X$ and a $\hat{y} \in F\hat{x}$ such that $\hat{y} \in T\hat{x}$. This completes our proof.

Remarks. 1. Note that the set of all $\hat{y}$ in (a) is a compact subset of $\operatorname{cl} F(X) \cap K$.

2. If $Z = X \times Y$ and $f = g = 1_{X \times Y}$, the identity function, then Theorem 5 becomes a generalization of the 1961 "geometric" lemma of Ky Fan [**25**, Lemma 4], [**28**, Lemma].

Particular forms. See Iohvidov [**47**, Theorem 1], Fan [**27**, Theorem 10], Browder [**15**, Theorem 17], Lassonde [**59**, Theorem 1.1′ and Proposition 1.7], and Park [**71**, Theorem 7]. For geometric forms, Fan [**25**], [**28**], [**29**, Theorem 2], [**30**, Theorem 10], Tan [**94**, Theorem 2], Shih and Tan [**85**, Theorem 3], Lin [**61**, Theorem 1], and Park [**74**, Theorem 9].

From Theorem 5, we have the following analytic alternative, which is a basis of various minimax inequalities.

Theorem 6. *Let X be a convex space, Y a Hausdorff space, $F : X \to ka(Y)$ an u.s.c. multifunction, $\alpha \geq \beta$, $f, g : X \times Y \to \overline{\mathbf{R}}$ extended real-valued functions, and K a nonempty compact subset of Y. Suppose that*

(6.1) *$g(x, y) \leq f(x, y)$ for all $(x, y) \in X \times Y$;*

(6.2) *for each $x \in X$, $\{y \in Y : g(x, y) > \alpha\}$ is compactly open ;*

(6.3) *for each $y \in F(X)$, $\{x \in X : f(x, y) > \beta\}$ is convex or empty ; and*

(6.4) *for each $N \in \mathcal{F}(X)$, there exists an $L_N \in kc(X)$ containing N such that, for each $x \in L_N \backslash F^+(K)$ and each $y \in Fx$, there exsits an $x_1 \in L_N$ satisfying $g(x_1, y) > \alpha$.*

Then either

(a) *there exsits a $\hat{y} \in \operatorname{cl} F(X) \cap K$ such that $g(x, \hat{y}) \leq \alpha$ for all $x \in X$, or*

(b) *there exists an $\hat{x} \in X$ and a $\hat{y} \in F\hat{x}$ such that $f(\hat{x}, \hat{y}) > \beta$.*

Proof. Put $Z = \overline{\mathbf{R}}$, $A = (\alpha, +\infty]$, and $B = (\beta, +\infty]$ in Theorem 5.

Particular forms. See Fan [**29**, Theorem 1], Brézis, Nirenberg, and Stampacchia [**12**, Theroem 1], Allen [**1**, Theorem 2], Ben-El-Mechaiekh, Deguire, and Granas [**8**, Théorème 2], [**9**, II, Théoremès 5.1 et 5.6, Corollaire 5.2], Tan [**94**, Theoerm 1], Shih and Tan [**83**, Theorem 1], [**86**, Theorem 3], Aubin and Ekeland [**4**, Theorem 6.3.9], Granas and Liu [**36**, Theorem 5.1], Lin [**61**, Theorem 5], Takahashi [**93**, Theorem 3], Ding and Tan [**21**, Theorem 1], Bae, Kim, and Tan [**5**, Theorem 1], and Park [**71**, Theorem 9], [**74**, Theorem 7].

As an another application of Theorem 5, we have the following generalized fixed point theorem.

Theorem 7. *Let X and C be nonempty convex subsets of a Hausdorff locally convex t.v.s. E. Let $F : X \to ca(X+C)$ be a compact u.s.c. multifunction. Suppose that one of the following conditions holds :*

(i) *X is closed and C is compact.*

(ii) *X is compact and C is closed.*

(iii) *$C = \{0\}$.*

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Proof. Let V be an open convex neighborhood of 0 in E, and Y a compact set such that $F(X) \subset Y \subset X + C$. Apply Theorem 5 to $A = B = C + V$, $Z = E$, and $f = g : X \times Y \to Z$ defined by $g(x, y) = y - x$ for $(x, y) \in X \times Y$. All of the requirements are satisfied. In fact,

(5.1) for each $x \in X$, $\{y \in Y : g(x, y) \in A\} = x + C + V$ is open ;

(5.2) for each $y \in Y$, $\{x \in X : g(x, y) \in A\} = y - C - V$ is convex ; and

(5.3) clearly holds since $Y = K$ is compact.

Moreover, Property (a) in Theorem 5 does not hold. In fact, since $Y \subset X + C$, for every $y \in Y$, there exists an $x \in X$ such that $y \in x + C + V$, that is, $y - x \in A$. Therefore Property (b) holds; that is, there exist an $x_V \in X$ and a $y_V \in Fx_V$ such that $y_V - x_V \in C + V$. In other words, we obtain the assertion :

(*) for each neighborhood V of 0 in E,

$$(F - i)(X) \cap (C + V) \neq \emptyset,$$

where $i : X \to E$ is the inclusion map.

(i) Since X is closed, so is $(F - i)(X)$. (See Lassonde [**59**, Lemma 2(i)].) Since C is compact and E is regular, (*) implies that $(F - i)(X) \cap C \neq \emptyset$; that is, there exists an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

(ii) Since $(F - i)(X)$ is compact and C is closed, the same conclusion follows as in Case (i).

(iii) If $C = \{0\}$, then $F : X \to ca(X)$ is a compact u.s.c. multifunction. Hence, F has a fixed point if and only if (*) with $C = \{0\}$ holds (Lassonde [**59**, Lemma 2(ii)]).

This completes our proof.

Particular forms. For $F : X \to cc(X + C)$, Theorem 7 reduces to Lassonde [**59**, Theorem 1.6 and Corollary 1.18], which generalize earlier results of Fan [**27**, Corollary to Theorem 10] and Browder [**15**, Theorem 19]. Theorem 7(iii) generalizes Himmelberg [**45**, Theorem 2], where it is assumed that $F : X \to cc(X)$. Particular forms of this result are due to Schauder [**81**, Satz II], Mazur [**62**], Bohnenblust and Karlin [**11**], Hukuhara [**46**], and Singbal [**90**], and, for a compact X, Brouwer [**13**, Theorem 4], Schauder [**81**, Satz I], Tychonoff [**99**, Satz], Kakutani [**53**, Theorem 1], Ky Fan [**24**], and Glicksberg [**34**, Theorem].

Remark. If X itself is compact in Theorem 7(iii), then X is an lc space and, hence, the conclusion follows from Begle's well-known theorem [**6**, Theorem 1].

7. An abstract variational inequality and a main minimax inequality

Theorem 1 can be used to obtain variational inequalities and minimax theorems. However, for the sake of simplicity, we adopt Corollary 1.1. The following abstract variational inequality is a typical one and has many applications.

Theorem 8. *Let X be a Hausdorff convex space, $p, q : X \times X \to \mathbf{R} \cup \{+\infty\}$, $h : X \to \mathbf{R} \cup \{+\infty\}$ functions, $F : X \to ka(X)$ an u.s.c. multifunction, and K a nonempty compact subset of X. Suppose that*

(8.1) $q(x, y) \le p(x, y)$ for $(x, y) \in X \times X$, and $p(x, y) + h(y) \le h(x)$ for all $x \in X$ and all $y \in Fx$;

(8.2) for each $x \in X$, $\{y \in X : q(x, y) + h(y) > h(x)\}$ is compactly open ;

(8.3) for each $y \in F(X)$, $\{x \in X : p(x, y) + h(y) > h(x)\}$ is convex or empty ; and

(8.4) for each $N \in \mathcal{F}(X)$, there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \backslash F^+(K)$ implies

$$Fx \subset \{y \in X : q(x_1, y) + h(y) > h(x_1) \text{ for some } x_1 \in L_N\}.$$

Then there exists a point $y_0 \in \operatorname{cl} F(X) \cap K$ such that

$$q(x, y_0) + h(y_0) \le h(x) \text{ for all } x \in X.$$

Moreover, the set of all solutions y_0 is a compact subset of $\operatorname{cl} F(X) \cap K$.

Proof. Define multifunctions $S, T : X \to 2^X$ by

$$Sx = \{y \in X : q(x, y) + h(y) > h(x)\},$$
$$Tx = \{y \in X : p(x, y) + h(y) > h(x)\}$$

for $x \in X$. Suppose that there exists a $y_0 \in \operatorname{cl} F(X) \cap K$ such that $y_0 \notin S(X)$. Then the conclusion follows. Therefore, we may assume that $\operatorname{cl} F(X) \cap K \subset S(X)$.

Then all of the requirements of Corollary 1.1 are satisfied. Hence, there exists an $x_0 \in X$ such that $Tx_0 \cap Fx_0 \neq \emptyset$. Let $y_0 \in Tx_0 \cap Fx_0$. Then $y_0 \in Fx_0$ and

$$p(x_0, y_0) + h(y_0) > h(x_0),$$

which contradicts (8.1). Moreover, the set of all solutions y_0 is the intersection

$$\bigcap_{x \in X} \{y \in \operatorname{cl} F(X) \cap K : q(x, y) + h(y) \leq h(x)\}$$

of compactly closed subsets of the compact set $\operatorname{cl} F(X) \cap K$. This completes our proof.

Remark. For $F = 1_X$, Theorem 8 includes the main results in [**75**], [**77**]. Therefore, as shown in [**75**], [**77**], the following are particular forms of Theorem 8.

Particular forms. See Brézis, Nirenberg, and Stampacchia [**12**, Application 3], Juberg and Karamardian [**51**, Lemma and Theorem 1], Mosco [**66**, Theorem 2.1], Allen [**1**, Corollaries 1 and 2], Takahashi [**91**, Theorem 3], Gwinner [**39**, Theorems 2 and 3], Lassonde [**59**, Proposition 1.4], and Park [**68**, Theorem 2], [**75**, Theorem 1], [**77**, Theorem 1].

The analytic alternative in Theorem 6 can be applied to the following main minimax inequality.

Theorem 9. *Let X be a convex space, Y a Hausdorff space, $F : X \to ka(Y)$ an u.s.c. multifunction, and K a nonempty compact subset of Y. Suppose that two functions $f, g : X \times Y \to \mathbf{R} \cup \{+\infty\}$ satisfy the following*

(9.1) *$g(x, y) \leq f(x, y)$ for all $(x, y) \in X \times Y$;*

(9.2) *for each $x \in X$, $y \mapsto g(x, y)$ is l.s.c. on any compact subset of Y;*

(9.3) *for each $y \in F(X)$, $x \mapsto f(x, y)$ is quasi-concave on X; and*

(9.4) *for each $N \in \mathcal{F}(X)$, there exists an $L_N \in kc(X)$ containing N such that, for each real λ, each $x \in L_N \backslash F^+(K)$, and each $y \in Fx$, there exists an $x_1 \in L_N$ satisfying $g(x_1, y) > \lambda$.*

Then (a) *there exists a $\hat{y} \in \operatorname{cl} F(X) \cap K$ such that*

$$\sup_{x \in X} g(x, \hat{y}) \leq \sup_{(x,y) \in F} f(x, y);$$

and

(b) *the following minimax inequality holds :*

$$\min_{y\in K}\sup_{x\in X} g(x,y) \le \sup_{(x,y)\in F} f(x,y).$$

Proof. It is clear that (a) implies (b). In order to show (a), we may assume that $\lambda = \sup\{f(x,y) : (x,y) \in F\} = \sup\{f(x,y) : x \in X \text{ and } y \in Fx\}$ is finite. Since Property (b) of Theorem 6 does not hold for $\lambda = \alpha = \beta$, we conclude that Property (a) of Theorem 6 holds. This completes our proof.

Particular forms. Theorem 9 is a far-reaching generalization of Fan's 1972 minimax inequality [**29**] and contains numerous extensions due to Fan [**29**, Theorem 1], [**32**, Theorem 5], Brézis, Nirenberg, and Stampacchia [**12**, Theorem 1], Takahashi [**91**, Lemma 1 and Theorem 2.1], [**93**, Theorems 4 and 6], Yen [**102**, Theorem 1], Aubin [**3**, Theorem 7.1.2], Ben-El-Mechaiekh, Deguire, and Granas [**8**, Corollaire 3], [**9**, II, Corollaire 5.3], Tan [**94**, Theorem 2], Shih and Tan [**83**, Theorem 2], [**84**, Theorem 2], Aubin and Ekeland [**4**, Theorem 6.3.9], Lassonde [**59**, Theorem 1.2 and Corollary 1.3], Granas and Liu [**36**, Théorème 3.2], [**37**, Theorem 7.1], Lin [**61**, Theorem 6], Ha [**41**, Theorem 1], and Park [**71**, Theorem 10]. In particular, Granas and Liu [**37**, Theorem 7.1] applied Theorem 9 to obtain a general version of the von Neumann minimax principle due to Sion and other minimax theorems due to Fan, Nikaido, and Kneser.

8. Other generalizations of the Ky Fan type geometric properties of convex spaces

In this section we obtain generalizations or variations of the 1961 geometric lemma of Fan [**25**, Lemma 4] which cannot be covered by our Theorem 5. We follow mainly Ha [**40**] and Takahashi [**91**].

We begin with the following consequence of Corollary 1.1.

Theorem 10. *Let K be a Hausdorff compact space, Y a convex space, and $A \subset B \subset C \subset K \times Y$ such that A is nonempty and closed in $K \times Y$. Suppose that*

(10.1) *for each $y \in Y$, $\{x \in K : (x,y) \in C\}$ is closed;*

(10.2) *for each $x \in K$, $\{y \in Y : (x,y) \notin B\}$ is convex or empty; and*

(10.3) *for each $y \in Y$, $\{x \in K : (x,y) \in A\}$ is acyclic.*

Then there exists a point $x_0 \in K$ such that $\{x_0\} \times Y \subset C$.

Proof. Suppose that for each $x \in K$ there exists a $y \in Y$ such that $(x,y) \notin C$. Define $S, T : Y \to 2^K$ by $Sy = \{x \in K : (x,y) \notin C\}$ and $Ty = \{x \in K : (x,y) \notin$

$B\}$ for $y \in Y$, and $F : Y \to 2^K$ by $Fy = \{x \in K : (x, y) \in A\}$ for $y \in Y$. Then Sy is open for $y \in Y$ and $\emptyset \neq \mathrm{co}\, S^{-}x \subset T^{-}x$ for $x \in K$. Further, Fy is acyclic for each $y \in Y$. Since A is closed in $K \times Y$, each Fy is closed in K and the graph of F is closed in $Y \times K$. Therefore, $F : Y \to ka(K)$ is a compact u.s.c. multifunction. Now, applying Corollary 1.1 with (Y, K, S, T, F, K) instead of (X, Y, S, T, F, K), we obtain a $y_0 \in Y$ such that $Ty_0 \cap Fy_0 \neq \emptyset$. For an $x_0 \in Ty_0 \cap Fy_0$, we have $(x_0, y_0) \notin B$ and $(x_0, y_0) \in A \subset B$, a contradiction.

Corollary 10.1. *Let X be a Hausdorff space, Y a convex space, and $A \subset B \subset C \subset X \times Y$ such that*

(1) *for each $y \in Y$, $\{x \in X : (x, y) \in C\}$ is compactly closed ; and*

(2) *for each $x \in X$, $\{y \in Y : (x, y) \notin B\}$ is convex or empty.*

Suppose that there exists a nonempty compact subset K of X and that A is closed in $X \times Y$ and

(3) *for each $y \in Y$, $\{x \in K : (x, y) \in A\}$ is acyclic.*

Then there exists an $x_0 \in K$ such that $\{x_0\} \times Y \subset C$.

Proof. Put $A_1 = A \cap (K \times Y)$, $B_1 = B \cap (K \times Y)$, and $C_1 = C \cap (K \times Y)$. Then apply Theorem 10 to (A_1, B_1, C_1).

Remark. If X is a convex space and if (3) is replaced by

(3)$'$ for each $y \in Y$, $\{x \in K : (x, y) \in A\}$ is nonempty and convex,

then Corollary 10.1 reduces to Ha [**40**, Theorem 3], which generalizes Fan's 1961 Lemma. In [**40**], [**41**], Ha used his result to obtain a minimax inequality and to prove Fan's non-separation theorem [**29**, Theorem 3] and other fixed point theorems. Note that in [**40**, Theorems 3 and 4], the convexity of X is superfluous.

Now Fan's 1961 Lemma can be stated as follows :

Corollary 10.2. *Let K be a compact convex space and $B \subset K \times K$ such that*

(1) *for each $y \in K$, $\{x \in K : (x, y) \in B\}$ is closed ;*

(2) *for each $x \in K$, $\{y \in K : (x, y) \notin B\}$ is convex or empty ; and*

(3) *$(x, x) \in B$ for each $x \in K$.*

Then there exists an $x_0 \in K$ such that $\{x_0\} \times K \subset B$.

Proof. Put $K = Y$, $A = \{(x, x) : x \in K\}$, and $B = C$ in Theorem 10. In this case the Hausdorffness assumption is dispensable by the argument of Section 5, since A is the graph of a single-valued continuous function.

Remark. Another type of generalization of Corollary 10.2 is possible (see Park [**70**], [**71**], [**74**]). It is well-known that Corollary 10.2 is equivalent to the Brouwer fixed point theorem, many of its generalizations, the Sperner lemma, the KKM theorem, the Fan-Browder fixed point theorem, Ky Fan's minimax inequality, and many others. (See Gwinner [**39**], Aubin and Ekeland [**4**], and Zeidler [**103**]).

We have other geometric properties which are closely related to Corollary 10.2.

Theorem 11. *Let K be a nonempty compact convex subset in a Hausdorff locally convex t.v.s. E and A an open subset of $K \times K$ such that*

(11.1) *$(x, x) \in A$ for every $x \in K$; and*

(11.2) *for each $x \in K$, $\{y \in K : (x, y) \notin A\}$ is acyclic or empty.*

Then there exists an $x_0 \in K$ such that $\{x_0\} \times K \subset A$.

Proof. Suppose the contrary. Then for any $x \in K$ there exists a $y \in K$ such that $(x, y) \notin A$. Define $F : K \to 2^K$ by $Fx = \{y \in K : (x, y) \notin A\}$ for each $x \in X$. Then each Fx is closed and acyclic. Note that $F : K \to ca(K)$ is u.s.c. Therefore, by Theorem 7(iii), F has a fixed point $x_0 \in K$, that is $(x_0, x_0) \notin A$. This contradicts (11.1).

Theorem 12. *If we replace condition* (11.2) *by*

(12.2) *for each $x \in K$, $\{y \in K : (x, y) \notin A\}$ is convex or empty,*

then Theorem 11 holds for a real t.v.s. E on which E^* separates points.

Proof. Just follow the proof of Theorem 11 and, instead of Theorem 7(iii), use Granas and Liu [**37**, Theorem 10.5] or Park [**68**, Theorem 6], [**69**, Theorem 5]. Those fixed point theorems are particular forms of Corollary 16.1 in the next section.

Remark. Theorems 11 and 12 generalize Takahashi [**91**, Theorem 12] and Browder [**15**, Theorem 17]. Takahashi gave two applications of his result to a Hausdorff locally convex t.v.s. However, in view of Theorem 12, those also hold for a real t.v.s. E on which E^* separates points.

9. Necessary and sufficient conditions for coincidences in topological vector spaces

Let E be a real Hausdorff t.v.s. We may assume that E^* contains nonzero elements and that E^* has any convex space structure.

From Corollary 1.1, we obtain the following :

Theorem 13. *Let X be a nonempty convex subset of a real Hausdorff t.v.s. E, K a nonempty compact subset of X, and $P, Q : X \to 2^E \backslash \{\emptyset\}$ multifunctions such that for each $f \in E^*$,*

(0) *$X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ is compactly closed.*

Then the following are equivalent :

(i) *There is an $\hat{x} \in K$ such that $\hat{x} \in \bigcap \{X_f : f \in E^*\}$.*

(ii) *There is an u.s.c. multifunction $F : E^* \to ka(X)$ satisfying*

(a) *$x \in Ff$ for $f \in E^*$ implies $x \in X_f$, and*

(b) *for each $N \in \mathcal{F}(E^*)$, there exists an $L_N \in kc(E^*)$ containing N such that $f \in L_N \backslash F^+(K)$ implies $Ff \subset \bigcup \{X \backslash X_g : g \in L_N\}$.*

Further, if either

(A) *E^* separates points of E and $P, Q : X \to kc(E)$, or*

(B) *E is locally convex, $P, Q : X \to cc(E)$, and one of Px and Qx is compact for each $x \in X$,*

then each of (i) *and* (ii) *are equivalent to the following :*

(iii) *There is an $\hat{x} \in X$ such that $P\hat{x} \cap Q\hat{x} \neq \emptyset$.*

Proof. (i) $\Longrightarrow$ (ii) Define $F : E^* \to \{\hat{x}\} \subset K$. Then all of the requirements in (ii) are trivially satisfied.

(ii) $\Longrightarrow$ (i) Define $S : E^* \to X$ by

$$Sf = \{x \in X : \sup f(Px) < \inf f(Qx)\} = X \backslash X_f$$

for $f \in E^*$. Then each Sf is compactly open by (0). Note that

$$S^- x = \{f \in E^* : \sup f(Px) < \inf f(Qx)\}$$

is convex for each $x \in X$. In fact, for any $r, s \geq 0$ with $r + s = 1$ and for any $f, g \in S^- x$, we have

$$\begin{aligned} \sup(rf + sg)(Px) &\leq r \sup f(Px) + s \sup g(Px) \\ &< r \inf f(Qx) + s \inf g(Qx) \\ &\leq \inf(rf + sg)(Qx). \end{aligned}$$

Now apply Corollary 1.1 with (E^*, X, S, S, F) instead of (X, Y, S, T, F). Then requirements (1) and (2) in Corollary 1.1 follow from the above observations and (4) from (b). However, (a) implies that S and F have no coincidence point. Hence, by Corollary 1.1, requirement (3) does not hold. Therefore, there exists an

$\hat{x} \in \operatorname{cl} F(E^*) \cap K \subset K$ which does not belong to $S(E^*)$; that is, $S^- \hat{x} = \emptyset$. Then $\hat{x} \in X_f$ for all $f \in E^*$.

Now suppose that either (A) or (B) holds.

(iii) $\Longrightarrow$ (i) For $y \in P\hat{x} \cap Q\hat{x}$, we have $\sup f(P\hat{x}) \geq fy \geq \inf f(Q\hat{x})$ for any $f \in E^*$. Therefore, $\hat{x} \in \bigcap\{X_f : f \in E^*\}$.

(i) $\Longrightarrow$ (iii) For $\hat{x}$ in (i), suppose that $P\hat{x} \cap Q\hat{x} = \emptyset$. Then each of (A) and (B) implies that there is an $f \in E^*$ such that $\inf f(Q\hat{x}) > \sup f(P\hat{x})$, by the standard separation theorems in a t.v.s. This contradicts $\hat{x} \in X_f$.

This completes our proof.

Remark. If P and Q are u.h.c., then (0) holds. For, the functions $x \mapsto \sup f(Px)$ and $x \mapsto \inf f(Qx)$ are u.s.c. and l.s.c., resp., on X. But, the converse is not true. An example of a pair (P, Q) which satisfies (0) but not u.h.c. can be given as follows :

Let $P, Q : [0, \infty) \to 2^{\mathbf{R}} \setminus \{\emptyset\}$ be given by

$$Px = \begin{cases} [1+x, 2+x] & \text{if } x \neq 1 \\ 1/2 & \text{if } x = 1, \end{cases}$$

$$Qx = \begin{cases} [-x-2, -x-1] & \text{if } x \neq 1 \\ -1/2 & \text{if } x = 1. \end{cases}$$

Theorem 14. *If X is compact in Theorem 13, then condition* (ii) *can be replaced by the following :*

(ii)′ *There is a multifunction $F : E^* \to ka(X)$ such that, for each polytope C in E^*, the restriction $F \,|\, C$ is u.s.c. and F satisfies* (a) *in* (ii).

Proof. (ii)′ $\Longrightarrow$ (i) can be shown as in (ii) $\Longrightarrow$ (i) using Corollary 1.2 instead of Corollary 1.1.

Remark. For u.d.c. multifunctions P and Q, Theorem 14 reduces to Granas and Liu [**37**, Theorem 10.1]. Moreover, they adopted a more restrictive condition than (ii)′.

From Theorem 14, we have a coincidence theorem for a class of multifunctions more general than u.h.c. ones.

Theorem 15. *Let X be a nonempty compact convex subset of a real Hausdorff t.v.s. E, and $P, Q : X \to 2^E$ multifunctions satisfying one of* (A) *and* (B) *in Theorem 13. Suppose that*

(15.1) *for each* $f \in E^*$, $X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ *is closed ; and*

(15.2) *for each* $(f, x) \in E^* \times X$, $fx = \min f(X)$ *implies* $x \in X_f$.

Then P *and* Q *have a coincidence point.*

Proof. Define $F : E^* \to 2^X$ by

$$Ff = \{x \in X : fx = \min f(X)\}$$

for $f \in E^*$. Then each Ff is nonempty, closed, and convex. In view of Theorem 14, it is sufficient to show that, for each polytope $C = \mathrm{co}\{f_1, f_2, \cdots, f_n\} \subset E^*$, $F\,|\,C$ has the closed graph in $C \times X$. Let (f_α, x_α) be a net in $C \times X$ with $x_\alpha \in Ff_\alpha$ converging to a point $(f_0, x_0) \in C \times X$. We have to show $x_0 \in Ff_0$. Let $\langle\ ,\ \rangle$ be an inner product for the finite dimensional subspace V of E^* generated by $f_1, f_2, \cdots, f_n$. By the Riesz representation theorem in $\mathbf{R}^n$, there exist $y_0, y_\alpha \in V$ such that, for all $f \in V$, we have $fx_0 = \langle f, y_0\rangle$ and $fx_\alpha = \langle f, y_\alpha\rangle$. Since V is of finite dimension, we have $y_\alpha \to y_0$ in V. Consequently,

$$\begin{aligned} f_0x_0 = \langle f_0, y_0\rangle &= \lim_\alpha \langle f_\alpha, y_\alpha\rangle \\ &= \lim_\alpha (\min f_\alpha(X)) \leq \min f_0(X), \end{aligned}$$

where the last inequality follows from the fact that min $f(X)$ as a function of f is u.s.c. (See Berge [**10**, p.80]). Thus $x_0 \in Ff_0$ and the proof is complete.

Remarks. 1. The above proof is a modification of that of Granas and Liu [**37**, Theorem 10.2].

2. Condition (15.2) is implied by one of the following :

(15.2)$'$ for each $(f, x) \in E^* \times X$ such that $fx = \min f(X)$, there exist $u \in Px$, $v \in Qx$ such that $fu \geq fv$.

(15.2)$''$ P is inward and Q is outward in the sense of Ky Fan [**29**]; that is, for each $(f, x) \in E^* \times X$ such that $fx = \min f(X)$, there exist $u \in Px$, $v \in Qx$ such that $fu \geq fx \geq fv$.

Particular forsm. Simons [**89**, Theorem 3.1] obtained Theorem 15(B) for u.h.c. multifunctions. Granas and Liu [**37**, Theorems 10.2 and 10.3] obtained particular cases of Theorem 15 for u.d.c. multifunctions satisfying (15.2)$'$ and (15.2)$''$, resp., instead of (15.2). Thus, Theorem 15 includes earlier works of Fan [**28**, Theorem 6], [**29**, Theorem 5], Lee and Tan [**60**, Theorem 4], and Takahashi [**92**, Theorem 11].

From Theorem 15, we have the following fixed point theorem for a class of multifunctions more general than weakly inward (outward) u.h.c. ones.

Theorem 16. *Let X be a nonempty compact convex subset of a real Hausdorff t.v.s. E, and R a multifunction satisfying either*

(A) *E^* separates points of E and $R : X \to kc(E)$; or*

(B) *E is locally convex and $R : X \to cc(E)$.*

(I) *Suppose that, for each $f \in E^*$,*

(16.1) *$\{x \in X : fx \geq \inf f(Rx)\}$ is closed ; and*

(16.2) *$d_f(Rx, \overline{I}_X(x)) = 0$ for every $x \in \operatorname{Bd} X$.*

Then R has a fixed point.

(II) *Suppose that, for each $f \in E^*$,*

(16.1)′ *$\{x \in X : \sup f(Rx) \geq fx\}$ is closed ; and*

(16.2)′ *$d_f(Rx, \overline{O}_X(x)) = 0$ for every $x \in \operatorname{Bd} X$.*

Then R has a fixed point. Further, if R is u.h.c., then $R(X) \supset X$.

Proof. (I) Put $Px = \{x\}$ and $Qx = Rx$ in Theorem 15. Since $I_X(x) = O_X(x) = E$ for $x \in \operatorname{Int} X$, (16.2) is equivalent to the following :

$$(*) \qquad d_f(Rx, \overline{I}_X(x)) = 0 \quad \text{for every } x \in X \text{ and } f \in E^*.$$

However, this implies (15.2) by Jiang [**48**, I, Lemma 2.1]. Therefore, by Theorem 15, R has a fixed point.

(II) Put $Px = Rx$ and $Qx = \{x\}$ in Theorem 15. The same argument as the above shows that R has a fixed point. For the surjectivity, let $y \in X$. Consider $Rx - y$ and apply Theorem 15 with $Px = Rx - y$ and $Qx = \{0\}$. Note that P is u.h.c. if R is, and hence, there exists an $x \in X$ such that $Rx \ni y$. This completes our proof.

Remarks. 1. If R is u.h.c., then (16.1) or (16.1)′ hold automatically, but not conversely, as shown by an example in the Remark following Theorem 13. If R is weakly inward or weakly outward, then (16.2) or (16.2)′, resp., holds automatically, but not conversely. For an example, see Jiang [**48**].

2. For a normed vector space E, note that (*) holds if and only if $d(Rx, \overline{I}_X(x)) = 0$, where d is the metric induced by the norm. This was given by Jiang [**48**, I, Corollary 3.6]

Particular forms. A different form of Theorem 16(B) is due to Simons [**87**, Theorem 2], which extends earlier works of Glebov [**33**, Theorem] and Cellina [**18**, Theorem 3]. See also Park [**76**]. Moreover, Park [**73**] applied Theorem 16 to obtain fixed point theorems for various types of condensing inward multifunctions.

The following consequence of Theorem 16 is well-known.

Corollary 16.1. *Let X be a nonempty compact convex subset of a real Hausdorff t.v.s. E, and $R : X \to 2^E$ an u.h.c. multifunction satisfying either* (A) *or* (B).

(I) *If R is weakly inward, then R has a fixed point.*

(II) *If R is weakly outward, then R has a fixed point and $R(X) \supset X$.*

Remark. Corollary 16.1 includes Lasry and Robert [**58**, Théorème 10], Cornet [**20**, Théorèmes 3.1 et 3.4] (see also [**3**, Theorems 15.1.3 and 15.1.4], [**4**, Theorems 6.4.14 and 6.4.15]), and numerous generalizations of the Brouwer or Kakutani fixed point theorems as follows :

Particular forms. See Brouwer [**13**], Schauder [**81**, Satz 1], Tychonoff [**100**, Satz], Rothe [**80**], Kakutani [**53**, Theorem 1], Glicksberg [**34**, Theorem], Fan [**24**], [**26**, Corollaire 2], [**28**, Theorem 3], [**29**, Theorem 6], [**32**, Corollary 3], Halpern [**42**], [**44**, Theorems 2, 3, and 5], Halpern and Bergman [**43**, Theorems 4.1 and 4.3], Browder [**14**, Theorems 1 and 2], [**15**, Theorems 3–5], Reich [**78**, Theorem 1.7], [**79**, Theorem 3.1], Kaczynski [**52**, Theorem 1], Arino, Gautier, and Penot [**2**, Theorem 1], Park [**68**], [**69**], [**76**], and Granas and Liu [**37**, Theorems 10.4 and 10.5].

Some of the major particular forms of Theorem 16 can be adequately summarized by the following enlarged version of the diagram given in [**68**].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex spaces, and IV for topological vector spaces having sufficiently many linear functionals. In fact, Theorem 16 contains all the results in the diagram.

111

References

1. G.Allen, *Variational inequalities, complementary problems, and duality theorems*, J. Math. Anal. Appl. **58** (1977), 1–10.
2. O.Arino, S.Gautier, and J.P.Penot, *A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations*, Funkcial Ekvac. **27** (1984), 273–279.
3. J.-P.Aubin, *Mathematical Methods of Game and Economic Theory*, North-Holland, Amsterdam, 1979.
4. J.-P.Aubin and I.Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, New York, 1984.
5. J.S.Bae, W.K.Kim, and K.K.Tan, *Another generalization of Ky Fan's minimax inequality and its applications*, Preprint.
6. E.G.Begle, *A fixed point theorem*, Ann. Math. **51** (1950), 544–550.
7. J.C.Bellenger, *Existence of maximizable quasiconcave functions on paracompact convex spaces*, J. Math. Anal. Appl. **123** (1987), 333–338.
8. H.Ben-El-Mechaiekh, P.Deguire, and A.Granas, *Une alternative non linéaire en analyse convexe et applications*, C. R. Acad. Sc. Paris **295** (1982), 257–259.
9. H.Ben-El-Mechaiekh, P.Deguire, and A.Granas, *Points fixes et coincidences pour les applications multivoque*, C. R. Acad. Sc. Paris **295** (1982), I, 337–340 ; II, 381–384.
10. C.Berge, *Espaces Topologiques*, Dunod, Paris, 1959.
11. H.F.Bohnenblust and S.Karlin, *On a theorem of Ville*, in "Contributions to the Theory of Games," Ann. of Math. Studies, No. 24, pp.155–160, Princeton Univ. Press, 1950.
12. H.Brézis, L.Nirenberg, and G.Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Un. Mat. Ital. **6** (1972), 293–300.
13. L.E.J.Brouwer, *Über Abbildungen von Mannigfaltigkeithen*, Math. Ann. **71** (1911/12), 97–115.
14. F.E.Browder, *A new generalization of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290.
15. F.E.Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.
16. F.E.Browder,*On a sharpened form of the Schauder fixed-point theorem*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 4749–4751.
17. F.E.Browder, *Coincidence theorems, minimax theorems, and variational inequalities*, Contemp. Math. **26** (1984), 67–80.

18. A.Cellina, *Fixed points of noncontinuous mappings*, Rend. Accad. Naz. Lincei **49** (1970), 30–33.
19. S.Y.Chang, *A generalization of KKM principle and its applications*, Soochow J. Math. **15** (1989), 7–17.
20. B.Cornet, *Paris avec handicaps et théorèmes de surjectivité de correspondances*, C. R. Acad. Sc. Paris **281** (1975), 479–482.
21. X.P.Ding and K.-K.Tan, *Generalized variational inequalities and generalized quasi-variational inequalities*, J. Math. Anal. Appl. **148** (1990), 497–508.
22. J.Dugundji and A.Granas,*KKM maps and variational inequalities*, Ann. Scuola Norm. Sup. Pisa **5** (1978), 679–682.
23. J.Dugundji and A.Granas, *Fixed Point Theory*, PWN, Warszawa, 1982.
24. Ky Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121–126.
25. Ky Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.
26. Ky Fan, *Sur un théorème minimax*, C. R. Acad. Sc. Paris **259** (1964), 3925–3928.
27. Ky Fan, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.
28. Ky Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 234–240.
29. Ky Fan,*A minimax inequality and applications*, in "Inequalities III" (O.Shisha, Ed.), pp.103–113, Academic Press, New York, 1972.
30. Ky Fan, *Fixed-point and related theorems for noncompact convex sets*, in "Game Theory and Related Topics" (O.Moeshlin and D.Pallaschke, Eds.), pp.151–156, North-Holland, Amsterdam, 1979.
31. Ky Fan, *A further generalization of Shapley's generalization of the Knaster-Kuratowski-Mazurkiewicz theorem*, in "Game Theory and Mathematical Economics" (O.Moeshlin and D.Pallaschke, Eds.), pp.275–279, North-Holland, Amsterdam, 1981.
32. Ky Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.
33. N.I.Glebov, *On a generalization of the Kakutani fixed point theorem*, Soviet Math. Dokl. **10** (1969), 446–448.
34. I.L.Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952),

170–174.

35. L.Górniewicz and A.Granas,*Some general theorems in coincidence theory*, I, J. Math. pures et appl. **60** (1981), 361–373.
36. A.Granas and F.-C.Liu, *Théorèmes du minimax*, C.R.Acad. Sc. Paris **298** (1984), 329–332.
37. A.Granas and F.-C.Liu, *Coincidences for set-valued maps and minimax inequalities*, J. Math. pures et appl. **65** (1986), 119–148.
38. A.Granas, *KKM-maps and their applications to nonlinear problems*, in "The Scottish Book" (R.D.Maulin, Ed.), pp.45–61, Birkhauser, Boston, 1982.
39. J.Gwinner, *On fixed points and variational inequalities — A circular tour*, Nonlinear Anal. TMA **5** (1981), 565–583.
40. C.-W.Ha, *Minimax and fixed point theorems*, Math. Ann. **248** (1980), 73–77.
41. C.-W.Ha, *On a minimax inequality of Ky Fan*, Proc. Amer. Math. Soc. **99** (1987), 680–682.
42. B.Halpern, *Fixed-point theorems for outward maps*, Doctoral Thesis, U.C.L.A., 1965.
43. B.R.Halpern and G.M.Bergman, *A fixed-point theorem for inward and outward maps*, Trans. Amer. Math. Soc. **130** (1968), 353–358.
44. B.R.Halpern, *Fixed point theorems for set-valued maps in infinite dimensional spaces*, Math. Ann. **189** (1970), 87–98.
45. C.J.Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.
46. M.Hukuhara, *Sur l'existence des points invariants d'une transformation dans l'espace fonctionnel*, Jap. J. Math. **20** (1950), 1-4.
47. I.S.Iohvidov, *On a lemma of Ky Fan generalizing the fixed point principle of A.N.Tihonov*, Soviet Math. Dokl. **5** (1964), 1523–1526.
48. J.Jiang, *Fixed point theorems for paracompact convex sets*, Acta Math. Sinica **4** (1988), I, 64–71; II, 234–241.
49. J.Jiang, *Fixed point theorems for convex sets*, Acta Math. Sinica **4** (1988), 356–363.
50. J.Jiang, *Coincidence theorems and minimax theorems*, Acta Math. Sinica **5** (1989), 307–320.
51. R.K.Juberg and S.Karamardian, *On variational type inequalities*, Boll. Un. Mat. Ital. **7** (1973), 336–338.
52. T.Kaczynski, *Quelques théorèmes de points fixes dans des espaces ayant suffisamment de functionnelles linéaires*, C.R.Acad. Sc. Paris **296** (1983), 873–874.

53. S.Kakutani, *A generalization of Brouwer's fixed-point theorem*, Duke Math. J. **8** (1941), 457–459.
54. O.P.Kapoor, *On an intersection lemma*, J. Math. Anal. Appl. **45** (1974), 354–356.
55. B.Knaster, C.Kuratowski, and S.Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.
56. H.-M.Ko and K.-K.Tan, *A coincidence theorem with applications to minimax inequalities and fixed point theorems*, Tamkang J. Math. **17** (1986),37–45.
57. H.Komiya, *Coincidence theorems and saddle point theorem*, Proc. Amer. Math. Soc. **96** (1986), 599–602.
58. J.-M.Lasry and R.Robert, *Degré pour les fonctional multivoques et applications*, C.R.Acad. Sc. Paris **280** (1975), 1435–1438.
59. M.Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.
60. C.-M.Lee and K.-K.Tan, *On Fan's extensions of Browder's fixed point theorems for multi-valued inward mappings*, J. Austral. Math. Soc. A**26** (1978), 169–174.
61. T.-C.Lin, *Convex sets, fixed points, variational and minimax inequalities*, Bull. Austral. Math. Soc. A**34** (1986), 107–117.
62. S.Mazur, *Un théorème sur les points invariants*, Ann. Soc. Polon. Math. **17** (1938), 110.
63. L.McLinden, *Acyclic multifunctions without metrizability*, in "Résumés des Colloque International: Théorie du Point Fixe et Applications," 5–9 juin 1989, Marseille–Luminy, pp.150–151.
64. G.Mehta, *Fixed points, equilibria and maximal elements in linear topological spaces*, Comm. Math. Univ. Carol. **28** (1987), 377–385.
65. G.Mehta and E.Tarafdar, *Infinite-dimensional Gale-Nikaido-Debreu theorem and a fixed point theorem of Tarafdar*, J. Econ. Th. **41** (1987), 333–339.
66. U.Mosco, *Implicit variational problems and quasi-variational inequalities*, in "Nonlinear Operators and the Calculus of Variations," Lect. Notes in Math. **543** (1976), 83–156.
67. J.von Neumann, *Über ein okonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebnisse eines Mathematischen Kolloquiums, **8** (1935), 73–83.
68. Sehie Park, *Fixed point theorems on compact convex sets in topological vector spaces*, Contemp. Math. **72** (1988), 183–191.
69. Sehie Park, *Fixed point theorems on compact convex sets in topological vector*

spaces, II, J. Korean Math. Soc. **26** (1989), 175–179.

70. Sehie Park, *Generalizations of Ky Fan's matching theorems and their applications*, J. Math. Anal. Appl. **141** (1989), 164–176.
71. Sehie Park, *Generalized Fan-Browder fixed point theorems and their applications*, Collection of Papers Dedicated to J. G. Park on His Sixtieth Birthday, pp.51–77, 1989.
72. Sehie Park, *Convex spaces and KKM families of subsets*, Bull. Korean Math. Soc. **27** (1990), 11–14.
73. Sehie Park, *Fixed points of condensing inward multifunctions*, J. Korean Math. Soc. **27** (1990), 185–192.
74. Sehie Park, *Generalizations of Ky Fan's matching theorems and their applications*, II, J. Korean Math. Soc. **28** (1991), 275–283.
75. Sehie Park, *Variational inequalities and extremal principles*, J. Korean Math. Soc. **28** (1991), 45–65.
76. Sehie Park, *A generalization of the Brouwer fixed point theorem*, Bull. Korean Math. Soc. **28** (1991), 43–47.
77. Sehie Park, *Remarks on some variational inequalities*, Bull. Korean Math. Soc. **28** (1991), 163–174.
78. S.Reich, *Fixed points in locally convex spaces*, Math. Z. **125** (1972), 17–31.
79. S.Reich, *Approximate selections, best approximations, fixed points, and invariant sets*, J. Math. Anal. Appl. **62** (1978), 104–113.
80. E.H.Rothe, *Zur Theorie de Topologischen Ordnung und der Vektorfelder in Banachshen Raumen*, Compositio Math. **5** (1937), 177–197.
81. J.Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. **2** (1930), 171–180.
82. S.Sessa,*Some remarks and applications of an extension of a lemma of Ky Fan*, Comm. Math. Univ. Carol. **29** (1988), 567–575.
83. M.-H.Shih and K.-K.Tan, *A further generalization of Ky Fan's minimax inequality and its application*, Studia Math. **78** (1983), 279–287.
84. M.-H.Shih and K.-K.Tan, *The Ky Fan minimax principle, sets with convex sections, and variational inequalities*, in "Differential Geometry–Calculus of Variations and Their Applications" (G.M.Rassias and T.M.Rassias, Eds.) pp.471–481, Marcel Dekker, New York, 1984.
85. M.H.Shih and K.K.Tan, *Generalized quasi-variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. **108** (1985), 333–343.
86. M.H.Shih and K.K.Tan, *A geometric property of convex sets with applications*

to minimax type inequalities and fixed point theorems, J. Austral. Math. Soc. A**45** (1988), 169–183.

87. S.Simons, *On a fixed-point theorem of Cellina*, Rend. Accad. Naz. Lincei **80** (1986), 8–10.
88. S.Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Proc. Symp. Pure Math. **45** (1986), Pt.2, 377–392.
89. S.Simons, *An existence theorem for quasiconcave functions with applications*, Nonlinear Anal. TMA **10** (1986), 1133–1152.
90. Singbal, *Generalized form of Schauder-Tychonoff's fixed-point principle*, in F.F. Bonsall, "Lectures on Some Fixed-Point Theorems of Functional Analysis," Mimeographed Notes, Tata Institute, Bombay, 1962.
91. W.Takahashi, *Nonlinear variational inequalitees and fixed point theorems*, J. Math. Soc. Japan **28** (1976), 168–181.
92. W.Takahashi, *Recent results in fixed point theory*, Southeast Asian Bull. Math. **4** (1980), 59–85.
93. W.Takahashi, *Fixed point, minimax, and Hahn-Banach theorems*, Proc. Symp. Pure Math. **45** (1986), 419–427.
94. K.K.Tan, *Comparison theorems on minimax inequalities, variational inequalities, and fixed point theorems*, J. London Math. Soc. **28** (1983), 555–562.
95. E.Tarafdar, *On nonlinear variational inequalities*, Proc. Amer. Math. Soc. **67** (1977), 95–98.
96. E.Tarafdar, *On minimax principles and sets with convex sections*, Publ. Math. Debrecen **29** (1982), 219–226.
97. E.Tarafdar, *Variational problems via a fixed point theorem*, Indian J. Math. **28** (1986), 229–240.
98. E.Tarafdar, *A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz Theorem*, J. Math. Anal. Appl. **128** (1987), 475–479.
99. E.Tarafdar and T.Husain, *Duality in fixed point theory of multivalued mappings with applications*, J. Math. Anal. Appl. **63** (1978), 371–376.
100. A.Tychonoff, *Ein Fixpunktsatz*, Math. Ann. **111** (1935), 767–776.
101. N.Yannelis and N.D.Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Economics, **12** (1983), 233–235.
102. C.L.Yen, *A minimax inequality and its applications to variational inequalities*, Pacific J. Math. **97** (1981), 477–481.

103. E.Zeidler, *Nonlinear Functional Analysis and its Applications*, 5 Vols., Springer–Verlag, New York/Berlin, 1986– .