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CYCLIC COINCIDENCE THEOREMS FOR ACYCLIC MULTIFUNCTIONS ON CONVEX SPACES

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1. Introduction

Usually, two multifunctions $S, T : X \multimap Y$ are said to have a coincidence if there exists a point $(x, y) \in X \times Y$ such that $y \in Sx \cap Tx$. On the other hand, for multifunctions $T : X \multimap Y$ and $U : Y \multimap X$, $(x, y) \in X \times Y$ is also called a *coincidence* of T and U if $y \in Tx$ and $x \in Uy$. In [**7**], Browder proved the existence of such coincidences in a variety of situations. In a recent paper [**17**], Simons extended Browder's result to the case of m (≥ 2) spaces, and obtained new results on cyclical coincidences.

The aim in this paper is to generalize all the main results of Simons [**17**] to much wider classes of multifunctions, for example, to acyclic valued multifunctions instead of convex valued ones. The basic tools that we use are the Brouwer fixed point theorem, some of its generalizations in [**1, 11, 13**], and the concept of a regular class of multifunctions in [**5, 11**]. Our new results are proved in a much simpler way than the proofs in [**17**], and contain many useful particular known cases.

2. Preliminaries

In this paper, multifunctions are always denoted by capital letters and single-valued functions are denoted by small letters.

For topological spaces X and Y, a multifunction $A : X \multimap Y$ is said to be *compact* if the range $A(X)$ is contained in a compact subset of Y. Recall that a nonempty space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A convex space X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls

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of its finite subsets [**12**]. This topology is called the *polytopology* in [**17**].

We introduce classes $\Phi, \mathbf{M}, \mathbf{K}$, and $\mathbf{V}$ of multifunctions $A : X \to Y$ as follows [**5, 11**]:

$A \in \Phi(X,Y) \iff$ (i) Y is a convex space; (ii) Ax is convex for each $x \in X$; and (iii) $\{\text{Int } A^{-1}y\}_{y\in Y}$ covers X, where Int denotes the interior with respect to the topology of X.

$A \in \mathbf{M}(X,Y) \iff$ (i) Y is a convex space; and (ii) for each nonempty compact subset K of X, there exist a finite subset $\{y_1, y_2, \ldots, y_k\}$ of Y and a continuous selection $s : K \to Y$ of $A|K$ such that $s(X) \subset P = \text{co}\{y_1, y_2, \ldots, y_k\}$.

$A \in \mathbf{K}(X,Y) \iff A$ is an u.s.c. multifunction with nonempty compact convex values.

$A \in \mathbf{V}(X,Y) \iff A$ is an u.s.c. multifunction with compact and acyclic values.

Note that the (poly)-Browder-Fan type multifunctions [**17**] belong to Φ, and the Kakutani type multifunctions [**17**] belong to $\mathbf{K}$. It is known that $\Phi \subset \mathbf{M}$ [4] and $\mathbf{K} \subset \mathbf{V}$ clearly.

Let $\mathcal{M}$ be a class of multifunctions. We say that the class $\mathcal{M}$ is *regular* [**5, 11**] provided that

(i) if $S \in \mathcal{M}$, then S has nonempty values;

(ii) given $S \xrightarrow{s} Y \xrightarrow{T} Z$ with s a continuous function and $T \in \mathcal{M}$, we have $T \cdot s \in \mathcal{M}$; and

(iii) given two functions $S : X \to Y$, $T : X' \to Y'$ in $\mathcal{M}$, their product $S \times T : X \times X' \to Y \times Y'$ is also in $\mathcal{M}$.

It is known that all of Φ, $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{V}$ are regular [**5, 11**].

Let $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$ with $(m-1)+1$ interpreted as 0. For a topological vector space E, E^* denotes its topological dual.

3. Main results

The following is our first cyclic coincidence theorem. This result will eventually be incorporated into Theorems 4 and 5.

THEOREM 1. *Let $m \geq 1$ and, for each $i \in \mathbf{Z}_m$, let $T_i \in \mathbf{M}(X_i, X_{i+1})$. Suppose that there exists an $i_0 \in \mathbf{Z}_m$ such that T_{i_0} is compact. Then there exists $(x_0, x_1, \ldots, x_{m-1}) \in X_0 \times X_1 \times \cdots \times X_{m-1}$ such that $x_{i+1} \in T_i x_i$ for all $i \in \mathbf{Z}_m$.*

Proof. Case 1 ($m = 1$). A direct consequence of the Brouwer fixed point theorem. The proof is semilar to, but simpler than, Case 2 discussed below.

Case 2 ($m \geq 2$). Without loss of generality, we suppose that T_{m-1} is compact; that is, $T_{m-1}(X_{m-1}) \subset K_0$ for some compact $K_0 \subset X_0$. Since $T_0 \in \mathbf{M}(X_0, X_1)$, $T_0|K_0$ has a continuous selection $f_0 : K_0 \to P_1$ where P_1 is a polytope; that is, the convex hull of some finite subset in X_1. Similarly, $T_i|P_i : P_i \to X_{i+1}$ has a continuous selection $f_i : P_i \to P_{i+1}$ for each $i = 2, \ldots, m-2$, where P_{i+1} is a polytope in X_{i+1}. Finally, $T_{m-1}|P_{m-1} : P_{m-1} \to X_0$ has a continuous selection $f_{m-1} : P_{m-1} \to P_0$, where P_0 is a polytope in K_0. Then the composition $f_{m-1}f_{m-2}\ldots f_0|P_0 : P_0 \to P_0$ has a fixed point $x_0 \in P_0$ by the Brouwer theorem. Now put $x_{i+1} = f_i x_i \in T_i x_i$ for $i = 0, 1, \ldots, m-2$. Then $f_{m-1}x_{m-1} = x_0 \in T_{m-1}x_{m-1}$. This completes our proof.

For the cases $m = 1$ and $m = 2$, Theorem 1 is due to Ben-El-Mechaiekh *et al.* [**5**, Théorèmes 2 et 3]. For Φ instead of $\mathbf{M}$, Theorem 1 is due to Simons [**17**, Theorem 1.4]. Moreover, a number of particular cases of Theorem 1 for Φ and applications have appeared in the literature as follows : For $m = 1$, Theorem 1 extends Browder [**6**, Theorem 1], Ben-El-Mechaiekh *et al.* [**2**, Théorème 1; **4**, Théorèmes 3.1 et 3.2], and Granas and Liu [**11**, Corollary 4.4]. Far-reaching generalizations are also given in [**15**]. For $m = 2$, Theorem 1 extends Ben-El-Mechaiekh *et al.* [**3**, Théorème 3; **4**, Théorème 4.3], and Granas and Liu [**11**, Corollary 4.5].

The second cyclic coincidence theorem of Simons [**17**, Theorem 2.5] can be generalized in two directions as follows :

THEOREM 2. *Let $k \geq 1$ and, for each $h \in \mathbf{Z}_k$, let Y_h be a nonempty compact convex subset of a topological vector space E_h on which E_h^* separates points, and $S_h \in \mathbf{K}(Y_h, Y_{h+1})$. Then there exists $(y_0, y_1, \ldots, y_{k-1}) \in Y_0 \times Y_1 \times \cdots \times Y_{k-1}$ such that $y_{h+1} \in S_h y_h$ for all $h \in \mathbf{Z}_k$.*

Proof. Case 1 ($k = 1$). For a nonempty compact convex subset of a topological vector space E on which E^* separates points, every $S \in \mathbf{K}(X, X)$ has a fixed point. This is due to Granas and Liu [**11**, Theorem 10.5] and Park [**13**, Theorem 6]. See also [**14**].

Case 2 ($k \geq 2$). Let $X = Y_0 \times \cdots \times Y_{k-1}$ and $E = E_0 \times \cdots \times E_{k-1}$

and define $S : X \to X$ by

$$S(y_0, \ldots, y_{k-1}) = S_{k-1}y_{k-1} \times S_0 y_0 \times \cdots \times S_{k-2}y_{k-2}$$

for $(y_0, \ldots, y_{k-1}) \in X$. Since $\mathbf{K}$ is regular, $S \in \mathbf{K}(X, X)$. Therefore, by Case 1, there exists $x = (y_0, \ldots, y_{k-1}) \in X$ such that $x \in Sx$. This completes our proof.

The case $k = 2$ of Theorem 2 is due to Granas and Liu [**11**, Theorem 12.1]. If all E_h are Hausdorff locally convex spaces, Theorem 2 reduces to Simons [**17**, Theorem 2.5], Theorem 2 for $k = 1$ to Fan [**8**] and Glicksberg [**9**], and for $k = 2$ to Browder [**7**, Theorem 1] and Granas and Liu [**10**, Théorème 5.1]. For finite dimensional spaces, the case $k = 2$ goes back to von Neumann [**18**].

Another generalization of [**17**, Theorem 2.5] is the following:

THEOREM 3. *Let $k \geq 1$ and, for each $h \in \mathbf{Z}_k$, let Y_h be a nonempty compact convex subset of a Hausdorff locally convex space E_h, and $V_h \in \mathbf{V}(Y_h, Y_{h+1})$. Then there exists $(y_0, y_1, \ldots, y_{k-1}) \in Y_0 \times Y_1 \times \cdots \times Y_{k-1}$ such that $y_{h+1} \in V_h y_h$ for all $h \in \mathbf{Z}_k$.*

Proof. Case 1 ($k = 1$). A nonempty compact convex subset X of a Hausdorff locally convex space is an *lc* space. Therefore, every $V \in \mathbf{V}(X, X)$ has a fixed point, by Begle [**1**, Theorem 1].

Case 2 ($k \geq 2$). Note that $\mathbf{V}$ is regular, and just follow the proof of Theorem 2.

Note that, for $\mathbf{K}$ instead of $\mathbf{V}$, Theorem 3 reduces to Simons [**17**, Theorem 2.5], and includes results in [**8, 9, 10, 11, 18**] as noted above.

As was the case with Theorem 1, Theorems 2 and 3 will be incorporated into Theorems 4 and 5, resp., as follows:

We now come to our general cyclic coincidence theorems.

THEOREM 4. *Let $m \geq 1$ and, for each $i \in \mathbf{Z}_m$, let either*

(1) $T_i \in \mathbf{M}(X_i, X_{i+1})$ or

(2) $T_i \in \mathbf{K}(X_i, X_{i+1})$ with X_{i+1} a compact subset in a topological vector space E_{i+1} on which E^*_{i+1} separates points.

Suppose that there exists an $i_0 \in \mathbf{Z}_m$ such that $T_{i_0} \in \mathbf{M}$ with X_{i_0} compact or $T_{i_0} \in \mathbf{K}$. Then there exists $(x_0, x_1, \ldots, x_{m-1}) \in X_0 \times X_1 \times \cdots \times X_{m-1}$ such that $x_{i+1} \in T_i x_i$ for all $i \in \mathbf{Z}_m$.

Proof. In view of Cases 1 of Theorems 1 and 2, we can suppose that $m \geq 2$ and, in view of Theorem 1, we may suppose that there exists $s \in \mathbf{Z}_m$ such that $T_s \in \mathbf{K}$. Let $s(0) < s(1) < \cdots < s(k-1)$ be exactly those values of $s \in \mathbf{Z}_m$ for which $T_s \in \mathbf{K}$. For each $h \in \mathbf{Z}_k$, let $Y_h = X_{s(h)+1}$ where $s(h)+1 \in \mathbf{Z}_m$. For each $h \in \mathbf{Z}_k$, we define $S_h : Y_h \to Y_{h+1}$ as follows :

Case 1 ($s(h+1) = s(h)+1 \in \mathbf{Z}_m$). Then $Y_{h+1} = X_{s(h)+2}$. Define $S_h = T_{s(h)+1} \in \mathbf{K}$.

Case 2 ($s(h+1) \neq s(h)+1 \in \mathbf{Z}_m$). Since each of $T_{s(h)+1}, \cdots, T_{s(h+1)-1}$ is $\mathbf{M}$ and $X_{s(h)+1}$ is compact, there exist continuous selections f_j for $j = s(h)+1, \ldots, s(h+1)-1$ as in the proof of Theorem 1 such that

$$X_{s(h)+1} \overset{f_{s(h)+1}}{\longrightarrow} P_{s(h)+2} \overset{f_{s(h)+2}}{\longrightarrow} P_{s(h)+3} \longrightarrow \cdots \overset{f_{s(h+1)-1}}{\longrightarrow} P_{s(h+1)},$$

where P_j's are polytopes in X_j, resp. Now let $S_h = T_{s(h+1)} \cdot f_{s(h+1)-1} \cdot \cdots \cdot f_{s(h)+2} \cdot f_{s(h)+1}$. Since $\mathbf{K}$ is regular, we know $S_h \in \mathbf{K}$ and $Y_{h+1} = X_{s(h+1)+1}$ is compact. Thus, from Theorem 2, there exists $(y_0, \ldots, y_{k-1}) \in Y_0 \times \cdots \times Y_{k-1}$ such that $y_{h+1} \in V_h y_h$. For all $h \in \mathbf{Z}_k$ we write $x_{s(h)+1} = y_h \in Y_h = X_{s(h)+1}$. If $r \in \mathbf{Z}_m \backslash \{s(h)+1 : h \in \mathbf{Z}_h\}$, then there exists $h \in \mathbf{Z}_k$ such that $r \in \{s(h)+2, \ldots, s(h+1)\}$. We write

$$x_r = f_{r-1} \cdot \cdots \cdot f_{s(h)+1}(x_{s(h)+1}) \in X_r.$$

Then $x_{i+1} \in T_i x_i$ for all $i \in \mathbf{Z}_m$. This completes our proof.

If each E_{i+1} is a Hausdorff locally convex space, then Theorem 4 reduces to Simons [**17**, Theorm 3.1]. The case with $m = 2$ for $T_0 \in \Phi(X, Y)$ and $T_1 \in \mathbf{K}(Y, X)$ generalizes Browder [**6**, Theorem 7; **7**, Theorem 3], Ben-El-Mechaiekh *et al.* [**3**, Théorème 1; **4**, Théorème 4.1].

The following is a somewhat simpler form of Theorem 4.

COROLLARY 4.1. *Let $m \geq 1$ and, for each $i \in \mathbf{Z}_m$, either*

(1) *$T_i \in \mathbf{M}(X_i, X_{i+1})$ and X_i a compact convex space, or*

(2) *$T_i \in \mathbf{K}(X_i, X_{i+1})$ and X_{i+1} a nonempty compact convex subset of a topological vector space E_{i+1} on which E^*_{i+1} separates points.*

For Φ instead of $\mathbf{M}$ and for Hausdorff locally convex spaces, Corollary 4.1 reduces to [**17**, Corollary 3.2]. Similarly, [**17**, Corollary 3.3] holds under weaker assumptions.

For Hausdorff locally convex spaces, Theorem 4 can be strengthened as follows :

THEOREM 5. *Let $m \geq 1$ and, for each $i \in \mathbf{Z}_m$, let X_i be convex spaces, and let either*

(1) *$T_i \in \mathbf{M}(X_i, X_{i+1})$ or*

(2) *$T_i \in \mathbf{V}(X_i, X_{i+1})$ with X_{i+1} a compact convex subset of a Hausdorff locally convex space E_{i+1}.*

Suppose that there exists an $i_0 \in \mathbf{Z}_m$ such that $T_{i_0} \in \mathbf{M}$ with X_{i_0} compact or $T_{i_0} \in \mathbf{V}$.

Then there exists $(x_0, x_1, \ldots, x_{m-1}) \in X_0 \times X_1 \times \cdots \times X_{m-1}$ such that $x_{i+1} \in T_i x_i$ for all $i \in \mathbf{Z}_m$.

Proof. Use Theorem 3 instead of Theorem 2 in the proof of Theorem 4.

For Φ and $\mathbf{K}$ instead of $\mathbf{M}$ and $\mathbf{V}$, resp., Theorem 5 reduces to Simons [**17**, Theorem 3.1]. Similarly, we can formulate generalized versions of [**17**, Corollaries 3.2 and 3.3] as was done for Theorem 4.

Added in Proof. Recently the author found that Theorems 3 and 5 also hold for topological vector spaces E on which E^* separates points.

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