Bull. Korean Math. Soc. **28**(1991), No. 1, pp. 33–37

A GENERALIZATION OF THE BROUWER FIXED POINT THEOREM

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The well-known Brouwer fixed point theorem states that a continuous function f from a nonempty compact convex subset of a Euclidean space into itself has a fixed point. The Brouwer theorem has numerous generalizations and applications. In the present paper, we give a generalization of the theorem for a broader class of functions $f : X \to E$, where X is a nonempty compact convex subset of a topological vector space E on which E^* separates points.

Our proof is based on the partition of unity argument and the following Fan–Browder fixed point theorem [2], [4]:

LEMMA (FAN–BROWDER). *Let X be a nonempty compact convex subset of a topological vector space E and T a multifunction on X such that, for each $x \in X$, Tx is a nonempty convex subset of X and, for each $y \in X$, $T^{-1}y = \{x \in X : y \in Tx\}$ is open in X. Then there is an $x_0 \in X$ such that $x_0 \in Tx_0$.*

It is well-known that Lemma is equivalent to the Brouwer theorem, the Sperner lemma, the Knaster–Kuratowski–Mazurkiewicz theorem, and many others.

For a (real or complex) topological vector space E, let E^* denote its dual space, that is, the vector space of all continuous linear functionals defined on E.

Let Bd, Int, and Cl denote the boundary, interior, and closure, resp., with respect to E. For any $X \subset E$ and $x \in E$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows :

$$I_X(x) = x + \bigcup_{\lambda>0} \lambda(X - x) \text{ and } O_X(x) = x + \bigcup_{\lambda<0} \lambda(X - x).$$

Received March 28, 1990.
Supported in part by the Basic Science Research Institute Program, Ministry of Education, 1989.

A function $f : X \to E$ is said to be *weakly inward* [*outward*, resp.] if $f(x) \in \text{Cl}\, I_X(x)$ [$\in \text{Cl}\, O_X(x)$, resp.] for each $x \in \text{Bd}\, X$.

The following is our main result :

THEOREM. *Let X be a nonempty compact convex subset of a topological vector space E on which E^* separates points, and $f : X \to E$ a weakly inward [outward] function such that*

$$\{x \in X : \text{Re}\, p(x) < \text{Re}\, p(f(x))\}$$

is open for all $p \in E^*$. Then f has a fixed point.

Proof. Suppose that $x \neq f(x)$ for each $x \in X$. Since E^* separates points of E, there exists a $p_x \in E^*$ such that $\text{Re}\, p_x(x) < \text{Re}\, p_x(f(x))$. Since $\{y \in X : \text{Re}\, p_x(y) < \text{Re}\, p_x(f(y))\}$ is open, there exists an open neighborhood U_x of x such that

$$\text{Re}\, p_x(u) < \text{Re}\, p_x(f(u)) \quad \text{for all} \quad u \in U_x.$$

Since X is compact and $\{U_x\}_{x \in X}$ is an open cover of X, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $\{U_{x_i}\}$ covers X. We may find a partition of unity corresponding this finite subcover of X, that is, a family of continuous functions $\alpha_1, \alpha_2, \ldots, \alpha_n : X \to [0,1]$ with the support of each α_i lying in U_{x_i} such that $\sum \alpha_i = 1$ on X. We form a function ϕ from X into the vector space of all continuous real functions on X by setting

$$\phi(x) = \sum_{i=1}^{n} \alpha_i(x)(\text{Re}\, p_{x_i}) \quad \text{for each} \quad x \in X.$$

If $\alpha_i(x) \neq 0$, then $x \in \text{Supp}\, \alpha_i \subset U_{x_i}$ and hence $\text{Re}\, p_{x_i}(x) < \text{Re}\, p_{x_i}(f(x))$. As $\alpha_i(x) \neq 0$ for at least one i, we have

$$(1) \qquad \begin{aligned} \phi(x)(x) &= \sum \alpha_i(x)(\text{Re}\, p_{x_i}(x)) \\ &< \sum \alpha_i(x)(\text{Re}\, p_{x_i}(f(x))) = \phi(x)(f(x)). \end{aligned}$$

Now, define a multifunction $T : X \to 2^X$ by

$$Tx = \{y \in X : \phi(x)(y - x) > 0\} \quad \text{for each} \quad x \in X.$$

Since $\phi(x)$ is $\mathbf{R}$–linear, that is, $\phi(x)(ry + sz) = r\phi(x)(y) + s\phi(x)(z)$ for all $y, z \in E$ and $r, s \in \mathbf{R}$, Tx is convex for each $x \in X$. Since $\alpha_i(x)$ and p_{x_i} are continuous, for each $y \in X$, $x \mapsto \phi(x)(y - x) = \sum \alpha_i(x)(\mathrm{Re}\ p_{x_i}(y - x))$ is continuous. Therefore, $T^{-1}y = \{x \in X : \phi(x)(y - x) > 0\}$ is open. Note that $x \notin Tx$ for each $x \in X$. Hence, by Lemma, there exists an $x_0 \in X$ such that Tx_0 is empty, that is,

$$\phi(x_0)(y) \leq \phi(x_0)(x_0) \quad \text{for all} \quad y \in X. \tag{2}$$

However, since $\phi(x_0) : X \to \mathbf{R}$ is continuous and $\mathbf{R}$–linear, (2) holds for all $y \in \mathrm{Cl}\ I_X(x_0)$. If $x_0 \in \mathrm{Int}\, X$, then $fx_0 \in E = \mathrm{Cl}\ I_X(x_0)$, and hence (2) holds for $y = fx_0$, which contradicts (1). If $x_0 \in \mathrm{Bd}\ X$, then we have the same contradiction. Therefore, f must have a fixed point.

For the outward case, if f is weakly outward, then the function $g : X \to E$ defined by $g(x) = 2x - f(x)$ is weakly inward and has the same fixed point with f. Note that $\{x \in X : \mathrm{Re}\ p(x) < \mathrm{Re}\ p(g(x))\} = \{x \in X : \mathrm{Re}\,(-p)(x) < \mathrm{Re}\,(-p)(f(x))\}$ for all $p \in E^*$.

This completes our proof.

REMARKS. 1. Any Hausdorff locally convex topological space and, hence, any normed vector space belongs to the class of topological vector spaces E on which E^* separates points.

2. The class of weakly inward functions $f : X \to E$ contains those satisfying $f(\mathrm{Bd}\ X) \subset X$ and self–functions $f : X \to X$.

3. Every continuous function $f : X \to E$ is weakly continuous, that is, $\mathrm{Re}\ pf : X \to \mathbf{R}$ is continuous for each $p \in E^*$. For every weakly continuous function $f : X \to E$, the set $\{x \in X : \mathrm{Re}\, p(x) < \mathrm{Re}\, p(f(x))\}$ is open for each $p \in E^*$. The converses are not ture.

4. The following examples show that even for a Euclidean space, our theorem properly generalizes the Brouwer theorem.

EXAMPLES. 1. Let $X = [0, 1]$ in $E = \mathbf{R}$, $fx = x$ for $x \in X \setminus (1/3,\ 2/3)$, and $fx = 1$ for $x \in (1/3,\ 2/3)$. Then the set $\{x \in X : p(x) < p(f(x))\}$ is open for all $p \in E^*$. Note that f is not weakly continuous.

2. Let $X = [0, 1]$ in $E = \mathbf{R}$. For a given $c \in (0, 1)$, let $f : X \to X$ be a function such that $f(x) > x$ for $x < c$ and $f(x) < x$ for $x > c$. Then c is a fixed point of f if and only if the set $\{x \in X : p(x) < p(f(x))\}$

is open for all $p \in E^*$. We can choose such an f which is not weakly continuous.

CONSEQUENCES. Historically well-known fixed point theorems due to Brouwer (1910), Schauder (1930), Tychonoff (1935), Ky Fan (1962, 1969), Halpern and Bergman (1968), Reich (1972), and Kaczynski (1983) for continuous functions are all consequences of our theorem. For the literature, see Park [5].

Moreover, we have the following:

COROLLARY 1. *Let X be a nonempty weakly compact convex subset of a topological vector space E on which E^* separates points, and $f : X \to E$ a weakly inward [outward] function such that*

$$\{x \in X : \operatorname{Re} p(x) < \operatorname{Re} p(f(x))\}$$

is weakly open for all $p \in E^*$. Then f has a fixed point.

COROLLARY 2. *Let X be a nonempty weakly compact convex subset of a metrizable locally convex topological vector space E, and $f : X \to E$ a weakly inward [outward] and weakly sequentially continuous function. Then f has a fixed point.*

Proof. For each weakly closed subset F of E, $f^{-1}(F)$ is sequentially closed in X, hence weakly compact by the Eberlein–Šmulian theorem as in [1], and $f^{-1}(F)$ is weakly closed. Therefore, f is weakly continuous. Hence, the conclusion follows from Corollary 1.

In case where $f : X \to X$, Corollary 2 reduces to Arino, Gautier, and Penot [1, Theorem 1]. It is open whether one can get rid of the metrizability.

From Corollary 1, we obtain the following :

COROLLARY 3. *Let X be a nonempty closed bounded convex subset of a reflexive Banach space E, and $f : X \to E$ a weakly inward [outward] function such that*

$$\{x \in X : \operatorname{Re} p(x) < \operatorname{Re} p(f(x))\}$$

is weakly open for all $p \in E^*$. *Then* f *has a fixed point.*

For a weakly sequentially continuous function $f : X \to X$, Corollary 3 reduces to Deimling [3, Ch.2, Exercise 10.4].

Finally, we note here that more general forms of our theorem for multifunctions defined on non-compact convex sets would be possible.

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