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VARIATIONAL INEQUALITIES AND EXTREMAL PRINCIPLES

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1. Introduction

In the 1930's, earlier works of Nikodym [15] and Mazur and Schauder [13] initiated the abstract approach to problems in calculus of variations. In 1966, Hartman and Stampacchia [9] proved the following remarkable result: for a continuous map f on a compact convex subset K of $\mathbf{R}^n$ into $\mathbf{R}^n$, there exists an $x_0 \in K$ such that $\langle fx_0, x - x_0\rangle \leq 0$ for all $x \in K$. They were also able to generalize their problem to a monotone function on a closed convex set in a reflexive Banach space.

Since the appearance of the Hartman-Stampacchia paper and one of Browder [4], such variational inequalities have received a great deal of attention and have been investigated and generalized in various points of views by a number of authors. The theory of variational inequalities has been used in a large variety of problems in nonlinear analysis, convex analysis, partial differential equations, mechanics, physics, optimization, and control theory.

Let E be a Hausdorff topological vector space (simply t.v.s. throughout this paper) and E^* its topological dual. Let us denote the pairing between E^* and E by $\langle w, x\rangle$ for $w \in E^*$ and $x \in E$. The typical variational inequality problem (VIP) is the following:

Given a nonempty set $X \subset E$, a function $f : X \to E^*$, and a real function $h : X \to \mathbf{R}$, find a point $x_0 \in X$ such that

$$\mathrm{Re}\langle fx_0, x_0 - y\rangle \leq h(y) - h(x_0) \quad \text{for all} \quad y \in X.$$

Such x_0 is called a solution of the VIP.

There are various types of generalizations or variations of such problem.

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Our purpose in this paper is to extend and unify basic results concerning variational inequalities and to give new and much simpler proofs of them. Our method is based on the generalized Fan-Browder fixed point theorem for noncompact convex spaces due to the author in his previous work [16]. Following the same line to [16], we obtain a number of new and generalized variational inequalities and their applications.

Section 2 is for preliminaries.

In section 3, we obtain sufficient conditions for the existence of solutions to abstract variational inequalities. Some basic results of Gwinner [8], Lassonde [10], Mosco [14] and Tan [21], and minimax inequalities due to Ky Fan [6], Brézis, Nirenberg, and Stampacchia [10], Allen [1], Takahashi [20], Tan [21] and Lin [11] are extended and unified.

Section 4 deals with the extremal principle of Mazur and Schauder [13]. Their principle is generalized and strengthened in various viewpoints.

2. Preliminaries

We follow mainly Lassonde [10] and our previous work [16].

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. In fact, if the convex space X is in a vector space E, we may regard that X has the relative finite topology. We recall that a set A in E is said to be *finitely open* if its intersection with any finite dimensional manifold M in E is open in the Euclidean topology of M. We note that any subset A of E open in a Hausdorff t.v.s. topology on E must be finitely open.

A nonempty subset L of a convex space X is called a *c-compact set* if for each finite subset $S \subset X$, there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. A subset A of a topological space Y is said to be *compactly closed* [resp. *open*] in Y if for every compact set $K \subset Y$ the set $A \cap K$ is closed [resp. open] in K.

A real-valued function $f : X \to \mathbf{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is open for each $r \in \mathbf{R}$; if X is a convex set in a vector space, then f is *quasi-concave* [resp. *quasi-convex*] whenever $\{x : X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is convex for each $r \in \mathbf{R}$.

For a subset K of a t.v.s. E and an $x \in E$, the *inward* and *outward sets* of K at x, $I_K(x)$ and $O_K(x)$, are defined as follows:

$$I_K(x) \equiv \{x + r(u - x) \in E : u \in K,\ r > 0\},$$
$$O_K(x) \equiv \{x - r(u - x) \in E : u \in K,\ r > 0\}.$$

Note that $K \subset I_K(x)$ and that if x is an internal point [5, p.410] of K, then $I_K(x) = O_K(x) = E$. Note also that every interior point is internal [5, p.413].

The boundary and interior will be denoted by Bd and Int, resp.

The following is due to the author [16] as a generalized Fan-Browder fixed point theorem :

THEOREM 0. *Let X be a convex space, Y a topological space, and $A, B : X \to 2^Y$ multifunctions satisfying the following:*

(a) *$Bx \subset Ax$ for each $x \in X$,*

(b) *$A^{-1}y$ is convex for each $y \in Y$,*

(c) *$B^{-1}y \neq \emptyset$ for each $y \in Y$,*

(d) *Bx is compactly open for each $x \in X$, and*

(e) *for some c-compact set $L \subset X$, the set $Y \backslash B(L)$ is compact.*

Then, for any continuous function $s : X \to Y$, there exists an $x_0 \in X$ such that $sx_0 \in Ax_0$.

3. Abstract variational and minimax inequalities

As a direct application of Theorem 0, we obtain the following sufficient conditions for the existence of solutions to an abstract variational inequality:

THEOREM 1. *Let X be a convex space, $p, q : X \times X \to \mathbf{R} \cup \{+\infty\}$ and $h : X \to \mathbf{R} \cup \{+\infty\}$ functions satisfying*

(i) *$q(x, y) \leq p(x, y)$ for $(x, y) \in X \times X$ and $p(x, x) \leq 0$ for all $x \in X$,*

(ii) *for each $y \in X$, $\{x \in X : p(x, y) + h(y) > h(x)\}$ is convex or empty,*

(iii) *for each $x \in X$, $\{y \in X : q(x, y) + h(y) > h(x)\}$ is compactly open, and*

(iv) *for some c-compact set $L \subset X$,*

$$K \equiv \{y \in X : q(x, y) + h(y) \leq h(x) \text{ for all } x \in L\}$$

is compact.

Then there exists a point $y_0 \in K$ *such that*

$$q(x, y_0) + h(y_0) \le h(x) \quad \text{for all} \quad x \in X.$$

Moreover, the set of all solutions y_0 *is a compact subset of* K.

Proof. Define multifunctions $A, B : X \to 2^X$ by

$$Ax \equiv \{y \in X : p(x, y) + h(y) > h(x)\},$$
$$Bx \equiv \{y \in X : q(x, y) + h(y) > h(x)\}$$

for $x \in X$. If $B^{-1}y_0 = \emptyset$ for some $y_0 \in X$, then the conclusion follows. Suppose that

(c) $B^{-1}y \neq \emptyset$ for each $y \in X$.

We also have the following:

(a) $Bx \subset Ax$ for each $x \in X$, by (i).

(b) $A^{-1}y$ is convex for each $y \in X$, by (ii).

(d) Bx is compactly open for each $x \in X$, by (iii).

(e) $K \equiv \{y \in X : y \notin Bx \text{ for all } x \in L\} = X \backslash B(L)$.

Therefore, by Theorem 0, there exists an $x_0 \in X$ such that $x_0 \in Ax_0$. However, this implies $p(x_0, x_0) + h(x_0) > h(x_0)$, which contradicts the condition (i). Moreover, the set of all solutions is the intersection

$$\bigcap_{x \in X} \{y \in K : q(x, y) + h(y) \le h(x)\}$$

of compactly closed subsets of the compact set K. This completes our proof.

REMARKS.

1. The conditions (ii) and (iii), resp., can be replaced by the following:

(ii)′ for each $y \in X$, $p(\cdot, y) - h(\cdot)$ is quasiconcave on X whenever $h(\cdot) < +\infty$.

(iii)′ for each $x \in X$, $q(x, \cdot) + h(\cdot)$ is l.s.c. on compact subsets of X.

2. In case when X is a closed convex subset of a t.v.s. E, the condition (iv) is implied by the following "coercivity condition":

(iv)′ a compact set $C \subset E$ and an $\overline{x} \in X \cap C$ can be found such that

$$q(\overline{x}, y) + h(y) > h(\overline{x}) \quad \text{for all} \quad y \in X \backslash C.$$

In fact, we can choose $L = \{\overline{x}\}$. Then our set $K \subset X \cap C$ is compact.

Therefore, for $p \equiv q$, Theorem 1 generalizes Mosco [14, Theorem 2.1] and Gwinner [8, Theorem 2].

COROLLARY 1.1. *Let X be a convex space, $h : X \to \mathbf{R} \cup \{+\infty\}$ a l.s.c. convex function, and $p, q : X \times X \to \mathbf{R} \cup \{+\infty\}$ functions satisfying*

(i) *$q(x, y) \leq p(x, y)$ for each $(x, y) \in X \times X$ and $p(x, x) \leq 0$ for all $x \in X$,*
(ii) *for each $y \in X$, $p(\cdot, y)$ is concave on X,*
(iii) *for each $x \in X$, $q(x, \cdot)$ is l.s.c. on compact subsets of X, and*
(iv) *for some c-compact set $L \subset X$,*

$$K \equiv \{y \in X : q(x, y) + h(y) \leq h(x) \quad \text{for all} \quad x \in L\}$$

is compact.

Then there exists a $y_0 \in K$ such that

$$q(x, y_0) + h(y_0) \leq h(x) \quad \text{for all} \quad x \in X.$$

Proof. Since the sum of two concave functions is concave, for each $y \in X$, $p(\cdot, y) - h(\cdot)$ is concave whenever $h(\cdot) < +\infty$. Since the sum of two l.s.c. functions is l.s.c., for each $x \in X$, $q(x, \cdot) + h(\cdot)$ is l.s.c. on compact subsets of X. Therefore, by Theorem 1, the conclusion follows.

REMARKS.

1. For $p \equiv q$, Corollary 1.1 is due to Lassonde [10, Proposition 1.4].
2. For $p \equiv q$, Corollary 1.1 improves Gwinner [8, Theorem 3]. In fact, Gwinner obtained his result under the stronger assumptions that $p = q$ is pseudo-monotone and that the coercivity condition (iv)′ in Remark 2 of Theorem 1, instead of (iv).

For $h \equiv 0$, Theorem 1 reduces to the following:

COROLLARY 1.2. *Let X be a convex space, and $p, q : X \times X \to \mathbf{R} \cup \{+\infty\}$ functions satisfying*

(i) $q(x,y) \leq p(x,y)$ *for* $(x,y) \in X \times X$ *and* $p(x,x) \leq 0$ *for all* $x \in X$,

(ii) *for each* $y \in X$, $\{x \in X : p(x,y) > 0\}$ *is convex or empty,*

(iii) *for each* $x \in X$, $\{y \in X : q(x,y) > 0\}$ *is compactly open, and*

(iv) *for some c-compact set* $L \subset X$,

$$K \equiv \{y \in X : q(x,y) \leq 0 \quad \text{for all} \quad x \in L\}$$

is compact.

Then there exists a $y_0 \in K$ *such that* $q(x, y_0) \leq 0$ *for all* $x \in X$. *Thus, in particular,*

$$\inf_{y \in K} \sup_{x \in X} q(x,y) \leq 0.$$

REMARKS.

1. The condition (iii) is implied by

(iii)′ for each $x \in X$, $q(x, \cdot)$ is l.s.c. on compact subsets of X.

In this case, the conclusion becomes

$$\min_{y \in K} \sup_{x \in X} q(x,y) \leq 0.$$

2. The condition (iv) is implied by

(iv)′ for some c-compact set $L \subset X$, we have $K \subset L$.

This fact was noted by Lin [11].

Tan [21, Theorem 1] is a particular form of Corollary 1.2 with (iii)′ and (iv)′ instead of (iii) and (iv), resp. Other results in Tan [21] can also be improved in the same way. See Remark 9 of Lin [11, p.116].

3. If X is a closed convex subset of a t.v.s. E, the coercivity condition (iv) can be replaced by

(iv)″ for some c-compact set L of E,

$$K \equiv \{y \in X : q(x,y) \leq 0\} \quad \text{for all} \quad x \in L \cap X\}$$

is compact.

Therefore, Corollary 1.2 improves Shih and Tan [18, Theorem 3]. Moreover, their assumption (d) is superfluous.

4. For $p \equiv q$, Corollary 1.2 generalize Aubin and Ekeland [2, Theorem 6.9], Takahashi [20, Theorem 2.1], and Allen [1, Theorem 2].

5. For $p \equiv q$, Corollary 1.2 improves the well-known result of Brézis, Nirenberg, and Stampacchia [3, Theorem 1]. If X is a closed convex subset of a t.v.s. E, instead of (iii) and (iv) in Corollary 1.2, the authors assumed the following:

(3) For any fixed $x \in X$, $p(x, y)$ is a l.s.c. function of y on the intersection of X with any finite dimensional subspace of E.

(5) There is a compact subset K of E and $x_0 \in K \cap X$ such that $p(x_0, y) > 0$ for all $y \in X \backslash K$.

Note that (3) implies (iii). For (3) implies that $\{y \in X : p(x,y) > r\}$ is finitely open in X for each $r \in \mathbf{R}$ and $x \in X$. Recall that X is equipped with the finite topology.

The condition (5) implies (iv). For, choose $L = \{x_0\}$. Then the set

$$\{y \in X : p(x_0, y) \leq 0\} \subset X \cap K,$$

because if $y \in X \backslash K$, then $p(x_0, y) > 0$, and hence compact.

Note also that their condition (4) is superfluous.

Similarly, in their application 2 in [3], the conditions (10) and (11) are superfluous.

6. If X is compact, then (iv) is satisfied automatically. In this case, for $p \equiv q$, Corollary 1.2 improves the Ky Fan minimax inequality [6].

COROLLARY 1.3. *Let X be a convex space and $p, q : X \times X \to \mathbf{R} \cup \{+\infty\}$ functions satisfying*

(i) *$q(x, y) \leq p(x, y)$ for each $(x, y) \in X \times X$.*

If $\alpha = \sup_{x \in X} p(x, x) < +\infty$, suppose also that

(ii) *for each $y \in X$, $\{x \in X : p(x, y) > \alpha\}$ is convex or empty,*

(iii) *for each $x \in X$, $\{y \in X : q(x, y) > \alpha\}$ is compactly open, and*

(iv) *for some c-compact set $L \subset X$,*

$$K \equiv \{y \in X : q(x, y) \leq \alpha \quad \text{for all} \quad x \in L\}$$

is compact.

Then we have

$$\inf_{y\in K}\sup_{x\in X} q(x,y) \le \sup_{x\in X} p(x,x).$$

Proof. If $\sup\limits_{x\in X} p(x,x) = +\infty$, the conclusion holds trivially. Suppose that $\sup_{x\in X} p(x,x) = \alpha < +\infty$. Applying Corollary 1.2 to $p(x,y)-\alpha$ and $q(x,y)-\alpha$ instead of $p(x,y)$ and $q(x,y)$, we have the conclusion.

REMARKS.

1. Corollaries 1.2 and 1.3 are actually equivalent.

2. If we replace (iii) by

(iii)′ for each $x \in X$, $q(x,\cdot)$ is l.s.c. on compact subsets of X, then the conclusion is

$$\min_{y\in K}\sup_{x\in X} q(x,y) \le \sup_{x\in X} p(x,y).$$

3. Note that Corollary 1.3 improves Lin [11, Theorem 6] and generalizes Tan [21, Theorem 2].

4. If X is a closed convex subset of a t.v.s. E, the condition (iv) is implied by

(iv)′ there exist a nonempty compact subset C of E and $x_0 \in X\cap C$, we have $q(x_0,y) > \alpha$ for all $y \in X\backslash C$.

In fact, we can choose $L = \{x_0\}$ and $K \subset X \cap C$.

Therefore, Corollary 1.3 extends Shih and Tan [18, Theorem 2].

5. For a compact X, Corollary 1.3 holds without assuming (iv). In this case, if we assume (iii)′ instead of (iii), then the conclusion is

$$\min_{y\in X}\sup_{x\in X} q(x,y) \le \sup_{x\in X} p(x,y),$$

since $\sup\limits_{x\in X} q(x,\cdot)$ is l.s.c. on X.

In this case, Corollary 1.3 improves Yen [22, Theorem 1].

6. For a compact X, if $p \equiv q$, Corollary 1.3 improves the Ky Fan minimax inequality [6] and Takahashi [20, Lemma 1].

COROLLARY 1.4. *Let H be a real Hilbert space, X a nonempty closed convex subset of H, and $a(\cdot,\cdot)$ a continuous bilinear form on H which is coercive (i.e., there is a constant $\gamma > 0$ such that $a(v,v) \geq \gamma\|v\|^2$ for all $v \in H$). Then for every $v' \in H^*$, there exists a unique vector $u \in X$ such that*

$$a(u, u - w) \leq \langle v', u - w\rangle \quad \textit{for all} \quad w \in X.$$

Proof. Apply Corollary 1.1 with H endowed with its weak topology, $h \equiv -v'$, and $p(v,w) \equiv q(v,w) \equiv a(v, v - w)$ for $v, w \in X$. Then all assumptions of Corollary 1.1 are satisfied. In particular, as to the coercivity condition (iv)′ in Remark 2 of Theorem 1, by taking w_0 to be a vector in $X \cap C$, where $C \equiv \{v \in H : \|v\| \leq R\}$ with $R > 0$ sufficiently large. In fact, since the form $a(\cdot,\cdot)$ is coercive, we have $[a(v, v - w_0) - \langle v', v - w_0\rangle] \to +\infty$ as $\|v\| \to \infty$. The uniqueness of the solution u follows from a standard argument. This completes our proof.

REMARKS.

1. Corollary 1.4 is due to Stampacchia [19] and Lions and Stampacchia [12]. Also see Stampacchia [19, Theorem 2.1], which is the key result in [19].
2. The following particular form of Corollary 1.4 is popular in the literature:

COROLLARY 1.5. *Let H be a real Hilbert space, X a nonempty closed convex subset of H, and $f : X \to H$ a continuous linear map such that, for a given $\alpha > 0$, we have $\langle fx, x\rangle \geq \alpha\|x\|^2$ for all $x \in H$. Then there exists a point $x_0 \in X$ such that*

$$\langle fx_0, y - x_0\rangle \geq 0 \quad \textit{for all} \quad y \in X.$$

Proof. Put $a(x,y) \equiv \langle fx, y\rangle$ and $v' \equiv 0$ in Corollary 1.4.

4. Extremal principles

In this section, we extend the key results in [17] which are generalizations of the well-known classic result of Mazur and Schauder [13] on minimum values.

From Theorem 1, we obtain the following:

THEOREM 2. *Let X be a convex space and $h : X \to \mathbf{R} \cup \{+\infty\}$ a function satisfying*

(a) *for each $y \in X$, $\{x \in X : h(y) > h(x)\}$ is convex or empty,*
(b) *for each $x \in X$, $\{y \in X : h(y) > h(x)\}$ is compactly open, and*
(c) *for some c-compact set $L \subset X$,*

$$K \equiv \{y \in X : h(y) \leq h(x) \quad \text{for all} \quad x \in L\}$$

is compact.

Then there exists a point $y_0 \in K$ such that

$$h(y_0) = \min h(X).$$

Proof. Put $p(x,y) \equiv q(x,y) = 0$ for $(x,y) \in X \times X$ in Theorem 1.

The set of such y_0 is called the *minimal set* of h.

COROLLARY 2.1. *Let X be a convex space and $h : X \to \mathbf{R} \cup \{+\infty\}$ a l.s.c. quasiconvex function satisfying the condition (c). Then h have a nonempty compact minimal set in K.*

REMARK. Corollary 2.1 is due to the author and S.K.Kim [17, Theorem 2], from which a generalization of well-known result of Mazur and Schauder [13] is obtained in [17].

Finally, we have the following.

COROLLARY 2.2. *Let X be a convex space in a vector space E, and $h : E \to \mathbf{R} \cup \{+\infty\}$ a convex function satisfying the condition (c) of Theorem 2 and that $h|X$ is l.s.c. Then there exists a $y_0 \in K$ such that*

$$h(y_0) \leq h(x) \quad \text{for all} \quad x \in I_X(y_0).$$

Proof. By Corollary 3.1, there exists $y_0 \in K$ such that $h(y_0) \leq h(x)$ for all $x \in X$. Suppose that there exists a $w \in I_X(x_0)\backslash X$ such that $h(w) < h(y_0)$. Then there exist $u \in X$ and $r > 1$ such that $w = y_0 + r(u - y_0)$, i.e.,

$$\frac{1}{r}w + (1 - \frac{1}{r})y_0 = u \in X.$$

Since h is convex,

$$h(u) \leq \frac{1}{r}h(w) + (1 - \frac{1}{r})h(y_0) < h(y_0),$$

a contradiction. Therefore, the conclusion holds.

REMARK. In Corollary 2.2, the set K is actually equal to the set

$$K' \equiv \{y \in X : h(y) \leq h(x) \quad \text{for all} \quad x \in I_L(y)\}.$$

In fact, we have $K' \subset K$ clearly. Conversely, if $h(y) \leq h(x)$ for all $x \in L$, then it holds for all $x \in I_L(y)$. This shows $K \subset K'$.

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