JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 146, 482–494 (1990)

On Generalizations of the Meir–Keeler Type Contraction Maps

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Submitted by C. Foias

Received March 18, 1988

In 1969, Meir and Keeler [29] obtained a remarkable generalization of the Banach contraction principle. Since then, there have appeared a number of generalizations of their result. In 1981, the second author and Bae [33] extended the Meir–Keeler theorem to two commuting maps by adopting Jungck's method. This influenced many authors, and, consequently, a number of new results in this line followed. Recent works of Sessa and others [46, 47] contain common fixed point theorems of four maps satisfying certain contractive type conditions.

In the present paper, we give a new result which encompasses most of such generalizations of the Meir–Keeler theorem. Further our result also includes many other generalizations of the Banach contraction principle.

Some authors have obtained their results on 2-metric spaces. However, 2-metric versions are easily obtained from metric ones by an obvious

* Supported by a grant from the Korea Science and Engineering Foundation in 1985–1986.

0022-247X/90 $3.00

modification. Therefore, for simplicity, we have confined this work to metric spaces.

Previous to this paper, Meir–Keeler type conditions have required continuity of the maps involved. In our theorem we remove this restriction. We also replace the condition of commutativity, or weakly commutative, by a weaker condition called compatible. As a consequence, our theorem is the most general fixed point result of its type and includes over 50 theorems in the literature as special cases.

Let (X, d) be a metric space, and A and S selfmaps of X. A and S are said to be weakly commuting at a point x if $d(ASx, SAx) \leqslant d(Sx, Ax)$. This property was first defined by Sessa [45] and is strictly weaker than the condition that A and S commute at x. A property weaker than that of weakly commuting is compatibility [20] or preorbitally commuting [54]. Two maps A and S compatible if, whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim Ax_n = \lim Sx_n = u$, then $\lim d(SAx_n, ASx_n) = 0$. Every weakly commuting map is compatible, but there are examples to show that the converse is false.

Our main result is the following.

THEOREM 1. *Let (X, d) be a complete metric space, and S, T selfmaps of X with S or T continuous. Suppose there exists a sequence $\{A_i\}$ of selfmaps of X satisfying*

(i) *either $A_i: X \to SX \cap TX$ for each i, or*

(i′) $S, T: X \to \bigcap_i A_i X$,

(ii) *each A_i is compatible with S and T,*

(iii) *each A_i weakly commutes with S at each point ξ for which $A_i\xi = S\xi$ and each A_i weakly commutes with T at each point η for which $A_i\eta = T\eta$, and*

(iv) *for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each $x, y \in X$,*

$$\varepsilon \leqslant M_{ij}(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(A_i x, A_j y) < \varepsilon,$$

where

$$M_{ij}(x, y) = \max\{d(Sx, Ty), d(Sx, A_i x), d(Ty, A_j y), [d(Sx, A_j y) + d(Ty, A_i x)]/2\}.$$

Then all the A_i, S, and T have a unique common fixed point.

For any $x_0 \in X$, in case (i) choose sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$y_1 = Sx_1 = A_1 x_0,\ y_2 = Tx_2 = A_2 x_1, \ldots,$$
$$y_{2n-1} = Sx_{2n-1} = A_{2n-1} x_{2n-2},\ y_{2n} = Tx_{2n} = A_{2n} x_{2n-1}, \ldots.$$

For case (i′) choose $\{x_n\}$ and $\{y_n\}$ as follows:

$$y_1 = A_1 x_1 = Sx_0, \qquad y_2 = A_2 x_2 = Tx_1, ...,$$
$$y_{2n-1} = A_{2n-1} x_{2n-1} = Sx_{2n-2}, \qquad y_{2n} x_{2n} = Tx_{2n-1}, ...$$

Define $d_n = d(y_n, y_{n+1})$.

LEMMA 1. $r = \inf_n d_n = 0$.

Proof. Assume (i) and assume $r > 0$. From (iv),

$$d_{2n} = d(y_{2n}, y_{2n+1}) = d(A_{2n} x_{2n-1}, A_{2n+1} x_{2n})$$
$$< M_{2n, 2n+1}(x_{2n-1}, x_{2n}) = d_{2n-1}.$$

Similarly $d_{2n+1} < d_{2n}$, and $\{d_n\}$ is monotone decreasing.

There exists a $\delta > 0$ such that (iv) is true for $\varepsilon = r > 0$. Choose N so that $n \geqslant N$ implies $r \leqslant d_n < r + \delta$. We may assume that n is odd. Since $M_{2n, 2n+1}(x_{2n-1}, x_{2n}) = d_{2n-1}$, $r \leqslant d_{2n-1} < r + \delta$. From (iv) this implies that

$$d_{2n} = d(y_{2n}, y_{2n+1}) = d(A_{2n} x_{2n-1}, A_{2n+1} x_{2n}) < r,$$

a contradiction. Therefore $r = 0$.

The proof using (i′) is similar.

LEMMA 2. *If there exists a point p such that $A_i \xi = S\xi = T\xi = p$ for each i, and each A_i weakly commutes with S and T at ξ, then p is the unique common fixed point of the A_i, S, and T.*

Proof. Suppose $Sp \neq p$. Since each A_i and S weakly commute at ξ, $d(A_i S\xi, SA_i \xi) \leqslant d(S\xi, A_i \xi) = 0$, and $A_i S\xi = SA_i \xi$. From (iv),

$$d(Sp, p) = d(SA_i \xi, A_j \xi) = d(A_i S\xi, A_j \xi) < M_{ij}(S\xi, \xi) = d(Sp, p),$$

a contradiction. Therefore $Sp = p$.

Suppose $Tp \neq p$. Since each A_i weakly commutes with T at ξ, $d(A_i T\xi, TA_i \xi) \leqslant d(T\xi, A_i \xi) = 0$, $A_i T\xi = TA_i \xi$. From (iv),

$$d(p, Tp) = d(A_i \xi, TA_j \xi) = d(A_i \xi, A_j T\xi) < M_{ij}(\xi, T\xi) = d(p, Tp),$$

a contradiction.

Therefore $Tp = p$ and $A_i p = A_i S\xi = Sp = p$, and p is a common fixed point of the A_i, S, and T.

Suppose that q is another common fixed point of the A_i, S, and T. If $p \neq q$, then, from (iv),

$$d(p, q) = d(A_i p, A_j q) < M_{ij}(p, q) = d(p, q),$$

a contradiction.

Proof of the Theorem. Let $x_0 \in X$ and choose $\{x_n\}$ and $\{y_n\}$ as in (i). By Lemma 1, $r = 0$.

Suppose that $d_{2n-1} = 0$ for some $n > 0$. Then $y_{2n-1} = y_{2n}$. Suppose that $y_{2n} \neq y_{2n+1}$. Then $M_{2n,2n+1}(x_{2n-1}, x_{2n}) \neq 0$, which implies, by the proof of Lemma 1, that $d_{2n} < d_{2n-1}$, a contradiction. Therefore $d_{2n} = 0$, and hence $d_k = 0$ for all $k \geqslant 2n-1$ and $\{y_n\}$ is Cauchy.

Similarly, the assumption that $d_{2n} = 0$ for some $n \geqslant 0$ implies that $d_k = 0$ all $k \geqslant 2n$ and, again, $\{y_n\}$ is Cauchy.

Suppose $d_n \neq 0$ for any n. By Lemma 1, $\{d_n\}$ monotone decreases to zero. We wish to show that $\{y_n\}$ is Cauchy. Suppose not. Then there exists an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. From (iv), there exists a δ satisfying $0 < \delta < \varepsilon$ for which (iv) holds. Since $\lim d(y_n, y_{n+1}) = 0$, there exists an N such that $m > N$ implies $d(y_m, y_{m+1}) < \delta/6$. Let $n_i \geqslant N$. We now show that there exists an integer j satisfying $n_i < j < n_{i+1}$ such that

$$\varepsilon + \frac{\delta}{3} \leqslant d(y_{n_i}, y_j) < \varepsilon + \frac{2\delta}{3}. \tag{$*$}$$

First of all, there exist values of j such that $d(y_{n_i}, y_j) \geqslant \varepsilon + \delta/3$. For example, choose $j = n_{i+1}$. For suppose that $d(y_{n_i}, y_{n_{i+1}-1}) < \varepsilon + \delta/3$. Then

$$\begin{aligned} d(y_{n_i}, y_{n_{i+1}}) &\leqslant d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) \\ &< \varepsilon + \delta/3 + \delta/6 < 2\varepsilon, \end{aligned}$$

a contradiction. The inequality is also true for $j = n_{i+1} - 1$. There are also values of j such that $d(y_{n_i}, y_j) < \varepsilon + \delta/3$. For example, choose $j = n_i + 1$ or $j = n_i + 2$. Without loss of generality we may assume that n_i is odd. Pick j to be the smallest even integer such that $j > n_i$ and $d(y_{n_i}, y_j) \geqslant \varepsilon + \delta/3$. Then $d(y_{n_i}, y_{j-2}) < \varepsilon + \delta/3$ and

$$\begin{aligned} d(y_{n_i}, y_j) &\leqslant d(y_{n_i}, y_{j-2}) + d(y_{j-2}, y_{j-1}) + d(y_{j-1}, y_j) \\ &\leqslant \varepsilon + \delta/3 + \delta/6 + \delta/6 = \varepsilon + 2\delta/3, \end{aligned}$$

and (*) is established. Note that

$$\begin{aligned}\varepsilon < \varepsilon + \delta/3 &\leqslant d(y_{n_i}, y_j) \\ &\leqslant \max\{d(y_{n_i}, y_j), d(y_{n_i}, y_{n_i+1}), d(y_j, y_{j+1}), \\ &\qquad [d(y_{n_i}, y_{j+1}) + d(y_j, y_{n_i+1})]/2\} = M_{n_i+1, j+1}(x_{n_i}, x_{j+1}).\end{aligned}$$

From the choice of j, $d(y_{n_i}, y_j) = d(Sx_{n_i}, Tx_{j+1}) < \varepsilon + 2\delta/3$, $d(y_{n_i}, y_{n_i+1}) < \delta/6$, $d(y_j, y_{j+1}) < \delta/6$, and

$$\begin{aligned}&[d(y_{n_i}, y_{j+1}) + d(y_j, y_{n_i+1})]/2 \\ &\quad \leqslant [d(y_{n_i}, y_j) + d(y_j, y_{j+1}) + d(y_j, y_{n_i}) + d(y_{n_i}, y_{n_i+1})]/2 \\ &\quad < \varepsilon + 2\delta/3 + \delta/6 < \varepsilon + \delta.\end{aligned}$$

Therefore, from (iv), we have $d(A_{n_i+1}x_{n_i}, A_{j+1}x_j) = d(y_{n_i+1}, y_{j+1}) < \varepsilon$. However,

$$\begin{aligned}d(y_{n_i}, y_j &\leqslant d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{j+1}) + d(y_{j+1}, y_j) \\ &< \delta/6 + \varepsilon + \delta/6 = \varepsilon + \delta/3,\end{aligned}$$

contradicting (*). Therefore, $\{y_n\}$ is Cauchy.

Since (X, d) is complete, there exists a $\xi \in X$ such that $\lim y_n = \xi$. Thus $\lim Sx_{2n-1} = \xi$ and $\lim Tx_{2n} = \xi$. Since $\lim Tx_{2n} = \xi$, $\lim A_{2n}x_{2n-1} = \xi$.

Assume that S is continuous. Then $\lim SA_{2n}x_{2n-1} = S\xi$;

$$\begin{aligned}d(A_{2n}Sx_{2n-1}, S\xi) &\leqslant d(A_{2n}Sx_{2n-1}, SA_{2n}x_{2n-1}) \\ &\qquad + d(SA_{2n}x_{2n-1}, S\xi) \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

since A_{2n} and S are compatible. Therefore $\lim A_{2n}Sx_{2n-1} = S\xi$.

Suppose $\xi \neq S\xi$. $\lim M_{2n,2n-1}(Sx_{2n-1}, x_{2n-1}) = d(S\xi, \xi)$. Choose $\varepsilon = d(S\xi, \xi)/2$. Then there exists a positive integer N_1 such that, for all $n \geqslant N_1$,

$$|M_{2n,2n-1}(Sx_{2n-1}, x_{2n-2}) - d(S\xi, \xi)| < \varepsilon,$$

i.e.,

$$\varepsilon = -\varepsilon + d(S\xi, \xi) < M_{2n,2n-1}(Sx_{2n-1}, x_{2n-2}) < d(S\xi, \xi) + \varepsilon.$$

Since $\lim d(A_{2n}Sx_{2n-1}, A_{2n-1}x_{2n-2}) = d(S\xi, \xi)$, there exists an integer $N_2 > N_1$ such that, for all $n \geqslant N_2$,

$$|d(A_{2n}Sx_{2n-1}, A_{2n-1}x_{2n-2}) - d(S\xi, \xi)| < \varepsilon/2;$$

i.e., using (iv),

$$-\varepsilon/2 + d(S\xi, \xi) < d(A_{2n}Sx_{n-1}, A_{2n-1}x_{2n-2}) < \varepsilon = d(S\xi, \xi)/2,$$

which implies $d(S\xi, \xi) < \varepsilon$, a contradiction.

Suppose $A_i\xi \neq \xi$ for some integer i. $\lim M_{i,2n}(\xi, x_{2n-1}) = d(\xi, A_i\xi)$. Choose $\varepsilon = d(\xi, A_i\xi)/2$. Then there exists a positive integer N_1 such that, for all $n \geqslant N_1$,

$$|M_{i,2n}(\xi, x_{2n-1}) - d(\xi, A_i\xi)| < \varepsilon;$$

i.e.,

$$\varepsilon = d(\xi, A_i\xi) - \varepsilon < M_{i,2n}(\xi, x_{2n-1}) < \varepsilon + d(A_i\xi, \xi).$$

Since $\lim d(A_i\xi, A_{2n}x_{2n-1}) = d(A_i\xi, \xi)$, there exists an integer $N_2 > N_1$ such that, for all $n \geqslant N_2$,

$$|d(A_i\xi, A_{2n}x_{2n-1}) - d(A_i\xi, \xi)| < \varepsilon/2;$$

i.e., using (iv),

$$-\varepsilon/2 + d(A_i\xi, \xi) < d(A_i\xi, A_{2n}x_{2n-1}) < \varepsilon = d(A_i\xi, \xi)/2,$$

which implies $d(A_i\xi, \xi) < \varepsilon$, a contradiction. Therefore $A_i\xi = \xi$ for each integer i.

Using (i) there exists a point $w \in X$ such that $A_i\xi = Tw = \xi$.

Suppose there exists an integer j such that $A_jw \neq \xi$. Then $M_{ij}(\xi, w) = d(\xi, A_jw)$. From (iv),

$$d(\xi, A_jw) = d(A_i\xi, A_jw) < d(\xi, A_jw),$$

a contradiction. Thus $A_jw = \xi$ for each j. From (iii) A_j and T weakly commute at w so that $d(A_jTw, TA_jw) \leqslant d(Tw, A_jw) = 0$. Therefore $T\xi = TA_jw = A_jTw = A_j\xi = \xi$. By Lemma 2, the A_i, S, and T have a unique common fixed point.

The proof, assuming the continuity of T, is similar, as are the cases under which (i′) is satisfied.

Since 1969 a number of extensions of the Meir–Keeler result have appeared. Most of them are consequences of our Theorem 1. We list some of them.

1. The necessity part of Theorem 3 of [47] is a special case of Theorem 1, with conditions (ii) and (iii) replaced by the weak commutativity of the A_i with S and T, and (iv) replaced with

$$d(A_ix, A_jy) \leqslant f(M_{ij}(x, y)),$$

where $f\colon \mathbb{R}^+ \to R^+$ which is nondecreasing and upper semicontinuous and satisfies $f(t) < t$ for each $t > 0$.

Since Theorem 1 includes the above result, it also includes the corresponding theorems in Chang [4]; Fisher [10–12]; Hadjic [14]; Khan and Imdad [25]; Khan and Fisher [26]; Kubiak [27]; Sessa *et al.* [46]; Singh [49]; Singh and Singh [50]; Singh and Tiwari [51]; and Yeh [57, 58].

The sufficiency part of Theorem 3 of [47] is trivial.

2. Theorem 1 includes the result of Kaneko [21], where each $A_i = f$; $S = g$; $T = k$; f commutes with g and k; f, g, k are continuous; and (iv) is replaced by $d(fx, fy) \leqslant hd(gx, ky)$, $0 \leqslant h < 1$.

It also includes Mukherjee [30], with each $A_i = g$, $S = T = f$, f is continuous and commutes with g, and (iv) is replaced by

$$d(gx, gy) \leqslant a_1 d(gx, fx) + a_2 d(gy, fy) + a_3 d(gx, fy) + a_4 d(gy, fx) + a_5 d(fx, fy),$$

where $a_i \geqslant 0$, $\sum_{i=1}^{5} a_i < 1$.

By interchanging x and y in the above definition and then adding, one obtains

$$d(gx, gy) \leqslant A(d(gx, fx) + d(gy, fy)) + B(d(gx, fy) + d(gy, fx)) + Cd(fx, fy),$$

where $A = (a_1 + a_2)/2$, $B = (a_3 + a_4)/2$, and $C = a_5$. Thus $2A + 2B + C = k < 1$, and this inequality in turn implies

$$d(gx, gy) \leqslant k \max\{d(gx, fx), d(gy, fy), [d(gx, fy) + d(gy, fx)]/2, d(fx, fy)\},$$

which is included in (iv).

Also included is Theorem 2 of Som and Mukherjee [52], where each $A_i = g$, $S = T = f$, f is continuous and commutes with g, and (iv) is replaced with

$$d(gx, gy) \leqslant \alpha d[\{d(fx, gx)\}^{1-\nu-\delta} \{d(fy, gy)\}^{\nu} \{d(fx, fy)\}^{\delta}],$$

where $0 \leqslant \alpha < 1$, $\nu + \delta < 1$, $\nu > 0$, $\delta \geqslant 0$.

If one replaces each of the distances in the above inequality with the maximum of three distances, then we obtain

$$d(gx, gy) \leqslant \alpha \max\{d(fx, gx), d(y, gy), d(fx, fy)\},$$

which is included in (iv).

3. With $S=T=1_x$, conditions (i)–(iii) are automatically satisfied. Theorem 1 then contains a number of fixed point theorems for families of maps, for example, Chatterji [7], with $A_i=f_0$, $A_j=f_n$, and (iv) replaced by

$$d(f_0x, f_n y) \leqslant q \max\{d(x, y), d(x, f_0x), d(y, f_n y), [d(x, f_n y) + d(y, f_0x)]/2\},$$

$n=1, 2, ..., 0<q<1$.

For Jaiswal and Singh [19], take $A_i=f_m$, $A_j=f_n$, and replace (iv) with

$$d(f_mx, f_n y) \leqslant \alpha d(x, y) + \beta[d(x, f_mx) + d(y, f_n y)] + \gamma[d(x, f_n y) + d(y, f_mx)],$$

$\alpha, \beta, \gamma \geqslant 0$, $\alpha+2\beta+2\gamma<1$.

The above inequality reduces to that of Chatterji with $q=\alpha+2\beta+2\gamma$.

Theorem 1 also includes Theorem 20 of Rhoades [41], and therefore includes Theorem 1 of Chang [4], Theorem 1 of Chatterjea [5], Theorem 1 of Hussian and Sehgal [16], Theorem 1 of Iseki [18], and Theorem 1 of Ray [38].

4. With $S=T=1_x$ and the sequence $\{A_i\}$ replaced by two maps f and g, Theorem 1 includes a number of results for pairs of maps.

In Bajaj [1], (iv) is replaced with

$$d(Sx, Ty) \leqslant \alpha \left[\frac{d(x, Sx)d(x, Ty) + d^2(x, y) + d(x, Sx)d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}\right]$$

for each $x, y \in X$ such that the denominator does not vanish, $0<\alpha<1$.

The above inequality clearly implies

$$d(Sx, Ty) \leqslant \alpha \max\{d(x, Sx), d(x, y)\}.$$

In Fisher [9] (iv) is replaced with

$$d(Sx, Ty) \leqslant c \frac{\{d(x, Sx)\}^2 + \{d(y, Ty)\}^2}{d(x, Sx) + d(y, Ty)}, \qquad 0<c<1,$$

provided $d(x, Sx) + d(y, Ty) \neq 0$, and $d(x, Sx) + d(y, Ty) = 0$ implies $d(Sx, Ty) = 0$.

For $d(x, Sx) + d(y, Ty) \neq 0$ the above inequality implies

$$d(Sx, Ty) \leqslant c \max\{d(x, Sx), d(y, Ty)\},$$

which remains valid when $d(x, Sx) + d(y, Ty) = 0$.

In Proposition 1 of Rao [37], (iv) is replaced with

$$d(T_1x, T_2y) \leqslant k_1 d(x, T_1x) + k_2 d(y, T_2y) + k_3 d(x, T_2y) + k_4 d(y, T_1x) + k_5 d(x, y),$$

$k_i \geqslant 0$, $k_1 + k_2 + 2k_3 + 2k_4 + k_5 < 1$.

The above inequality implies

$$\begin{aligned} d(T_1x, T_2y) &\leqslant (k_1 + k_2) \max\{d(x, T_1x), d(y, T_2y)\} \\ &\quad + 2(k_3 + k_4)[d(x, T_2y) + d(y, T_1x)]/2 + k_5 d(x, y) \\ &\leqslant q \max\{d(x, T_1x), d(y, T_2y), [d(x, T_2y) \\ &\quad + d(y, T_1x)]/2, d(x, y)\}, \end{aligned} \tag{1}$$

where $q = k_1 + k_2 + 2(k_3 + k_4) + k_5$.

In Theorem 1 of Reilly [40] (iv) is replaced with

$$d(fx, gy) \leqslant ad(x, y) + b[d(x, fx) + d(y, gy)] + c[d(x, gy) + d(y, fx)],$$

where $a, b, c \in R$ and satisfy $a + 2c < 1$, $b + c < 1$, $c \geqslant 0$, $0 \leqslant (a + b + c)/(1 - b - c) < 1$.

Note that $(a + b + c)/(1 - b - c) < 1$ implies $a + 2b + 2c < 1$, and the contractive definition implies

$$d(fx, gy) \leqslant k \max\{d(x, fx) + d(y, gy)]/2, [d(x, gy) + d(y, fx)]/2\}, \tag{2}$$

where $k = a + 2b + 2c$. Equation (2) is a special case of (1).

In Rus [43], (iv) is replaced with

$$d(fx, gy) \leqslant \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)],$$

where $\alpha, \beta, \gamma \geqslant 0$, $\alpha + 2\beta + 2\gamma < 1$, which is included in (2).

In Pachpatte [31], (iv) is replaced with

$$d(Sx, Ty) \leqslant q \max\left[\frac{d(x, Sx) d(x, Ty)}{d(x, Ty) + d(y, Sx)}, \frac{d(y, Ty) d(y, Sx)}{d(x, Ty) + d(y, Sx)}\right]$$

for all x, y such that $d(x, Ty) + d(y, Sx) \neq 0$, $0 < q < 1$. The above inequality implies

$$\begin{aligned} d(Sx, Ty) &\leqslant q \max\{d(x, Sx), d(y, Ty)\} \\ &\leqslant q \max\{d(x, Sx), d(y, Ty), [d(x, Ty) + d(y, Sx)]/2 \end{aligned}$$

which is also valid for $d(x, Ty) + d(y, Sx) = 0$ and is included in (1).

In Sharma and Bajaj [48], (iv) is replaced with

$$d(Sx, Ty) \leqslant \beta \frac{d(x, Sx)d(x, Ty) + d(y, Sx)d(y, Ty)}{d(x, Sx) + d(x, Ty)}$$

for all x, y such that $d(x, Sx) + d(x, Ty) \neq 0$ and $0 < \beta < \frac{1}{2}$.

For all x, y such that $d(x, Sx) + d(x, Ty) \neq 0$, the above inequality implies that

$$\begin{aligned} d^2(Sx, Ty) &\leqslant \beta[d(x, Sx)d(x, Ty) + d(y, Sx)d(y, Ty)] \\ &\leqslant 2\beta \max\{d(x, Sx), d(y, Ty)\}[d(x, Ty) + d(y, Sx)]/2 \\ &\leqslant 2\beta \max\{d(x, Sx)[d(x, Ty) + d(y, Sx)]/2, d(y, Ty)[d(s, Ty) \\ &\quad + d(y, Sx)]/2\} \\ &\leqslant 2\beta \max\{d^2(x, Sx), d^2(y, Ty), [d(x, Ty) + d(y, Sx)]^2/2\}, \end{aligned}$$

which implies that S and T satisfy (1) with $q = \sqrt{2\beta}$.

Definition (1) is definition (146) of [41]. Therefore Theorem 14 of [41] is also a special case of Theorem 1. Other included results are Sehgal [44] and Srivastava and Gupta [53].

5. With $S = T = 1_x$, Theorem 1 provides a more general Meir–Keeler type theorem for a pair of maps f and g, since the restriction of continuity of f and g has now been removed. Thus our theorem generalizes our previous Theorem 2 in [35], as well as the corresponding Meir–Keeler type results in Ganguly [13], Hwang [17], Pant [32], Park and Bae [33], Park and Moon [34], Rhoades [41], and Yen and Chung [57].

6. With $S = T = 1_x$ and each $A_i = T$, Theorem 1 includes a number of results for single maps. For those of Meir–Keeler type Theorem 1 includes Meir and Keeler [29], Maiti and Pal [28], and Rao and Rao [36].

7. For single maps our Meir–Keeler theorem reduces to a generalization of (21) of [41]. Therefore it also includes the corresponding theorems in Chatterjea [6], Ciric [8], Hardy and Rogers [15], Kannan [22], Khan [23], Reich [39], and Zamfirescu [58].

8. Condition (iv) is implied by contractive conditions given by a gauge function and thus includes results of Boyd and Wong [2], Browder [3], and Khan and Imdad [24].

9. For a non-complete metric space, the following version of Theorem 1 is true, with essentially the same proof.

THEOREM 1′. *Let (X, d) be a metric space, with S, T, and $\{A_i\}$ the same as in Theorem* 1 *and satisfying* (i) *or* (i′), (ii), (iii), *and* (iv). *If, in addition,* *the sequence* $\{y_n\}$ *has a convergent subsequence, then the conclusions of Theorem* 1 *are true.*

By interchanging the roles of S and T with A_i and A_j in Theorems 1 and 1′ we obtain the following.

THEOREM 2. *In Theorems* 1 *and* 1′, (iv) *can be replaced by the following, without affecting the conclusions*:

(iv′) *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\varepsilon \leqslant M'_{ij}(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Sx, Ty) < \varepsilon,$$

where

$$M'_{ij}(x, y) = \max\{d(A_i x, A_j y), d(A_i x, Sx), d(A_j y, Ty), [d(A_i x, Ty) + d(A_j y, Sx)]/2\}.$$

ACKNOWLEDGMENT

The authors take this opportunity to thank Brian Fisher for the proof, in Theorem 1, that $d_n = 0$ for some n implies $d_k = 0$ for all $k \geqslant n$.

Note added in proof. Preprints of this paper were circulated prior to its publication. As a result, Gerald Jungck has kindly pointed out that hypothesis (iii) of Theorem 1 is not needed, since it follows immediately from the fact that compatible maps commute at coincidence points. (See, e.g., Proposition 2.2 of [20].)

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