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ON GENERALIZED EXTREMAL PRINCIPLES

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In this paper we obtain generalized and strengthened versions of the extremal principle of S. Mazur and J. Schauder [5]. We use the following generalization of the existence theorem of minimizable quasiconvex functions on convex spaces due to Ky Fan [2, Theorem 8].

THEOREM 1. *Let X be a convex space, and Φ a nonempty convex set of l.s.c. quasiconvex real functions on X. Let S be a subset of $X\times\Phi$ such that*

(a) *for each $\phi\in\Phi$, $S(\phi)=\{x\in X : (x,\phi)\in S\}$ is open in X, and*

(b) *for each $x\in X$, $S(x)=\{\phi\in\Phi : (x,\phi)\in S\}$ is convex and nonempty.*

Then either there exists a $(y_1,\phi_1)\in S$ such that $\phi_1(y_1)=\min \phi_1(X)$, or, for any c-compact subset L of X and nonempty compact subset K of X, there exists a $(y_2,\phi_2)\in S$ such that

$$y_2\in X\setminus K \text{ and } \phi_2(y_2)\leq\inf \phi_2(L).$$

Theorem 1 is due to the first author and J.S. Bae [6, Theorem 2]. Note that a generalization of Fan [2, Theorem 8] is given by J.C. Bellenger [1]. Recently, in [6, Theorem 1], the paracompactness assumption on X in [1] is removed.

In Theorem 1, a *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets [4]. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite subset $S\subset X$, there is a compact convex subset $L_S\subset X$ such that $L\cup S\subset L_S$ [4].

A real-valued function $f : X\rightarrow \boldsymbol{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x\in X : fx>r\}$

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[resp. $\{x \in X : fx < r\}$] is open for each $r \in \boldsymbol{R}$; if X is a convex set in a vector space, then f is *quasiconcave* [resp. *quasiconvex*] whenever $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is convex for each $r \in \boldsymbol{R}$.

From Theorem 1, we have the following:

THEOREM 2. *Let X be a convex space and $h : X \to \boldsymbol{R}$ a l.s.c. quasiconvex function such that for some c–compact subset L of X,*

$$K = \{y \in X : h(y) \leq \inf\ h(L)\}$$

is nonempty and compact. Then h has a nonempty compact minimal set $\{y_0 \in X : h(y_0) = \min\ h(X)\}$ in K.

Proof. Put $\Phi = \{h\}$ and $S = \{(x, h) : x \in X\} = X \times \Phi$ in Theorem 1. Since (a) and (b) hold automatically, by Theorem 1, h has a minimal point $y_0 \in K$. Since the minimal set is the intersection

$$\bigcap_{x \in X} \{y \in K : h(y) \leq h(x)\}$$

of closed subsets of the compact set K, it is compact.

Now we consider reflexive Banach spaces.

THEOREM 3. *Let X be a nonempty convex set in a reflexive Banach space E and $h : X \to \boldsymbol{R}$ a l.s.c. quasiconvex function satisfying the following coercivity condition*:

(*) *for some nonempty closed bounded convex subset L of X, the set*

$$K = \{y \in X : h(y) \leq \inf h(L)\}$$

is nonempty closed bounded.

Then h attains its minimum at some $y_0 \in K$.

Proof. Let us switch to the weak topology. For any $r \in \boldsymbol{R}$, the set $\{x \in X : h(x) \leq r\}$ is closed and convex, hence weakly closed. This implies that h is weakly l.s.c. Further, L is weakly c–compact and K is weakly compact. Therefore, by Theorem 2, the conclusion follows.

From Theorem 3, we have the following well-known result of Mazur and Schauder [5].

COROLLARY 1. *Let X be a nonempty closed convex set in a reflexive Banach space X and $h: X \to \boldsymbol{R}$ a l. s. c. quasiconvex and coercive (i. e., $|h(x)| \to \infty$ as $\|x\| \to \infty$) function. If h is bounded from below, then h attains its minimum at some $y_0 \in X$.*

Proof. It suffices to show that coerciveness implies (*) in Theorem 3. Let $d = \inf h(X)$. Then we can find $\rho > 0$ such that $L = B(0, \rho) \cap X \neq \phi$ and $h(y) > d+1$ for all $y \in X \backslash L$, where B denotes the closed ball. Note that

$$K = \{y \in X : h(y) \leq \inf h(L)\} \subset L,$$

and hence K is bounded.

The following example shows that Theorems 2 and 3 properly generalize Corollary 1.

EXAMPLE. Let $X = \boldsymbol{R}$, $L = [0, 1]$, and $K = [1, 2]$. For any $a \in \boldsymbol{R}$, define $h: X \to \boldsymbol{R}$ by

$$h(x) = \begin{cases} a & \text{if } x < 1 \\ a-1 & \text{if } 1 \leq x \leq 2 \\ a & \text{if } x > 2. \end{cases}$$

Clearly, h is l. s. c. and quasiconvex. Note that h satisfies all the requirements of Theorems 2 and 3. However, h is not coercive, and hence Corollary 1 is not applicable.

If X is bounded in Corollary 1, then the coercivity condition is satisfied automatically. Hence, we have

COROLLARY 2. *Let X be a closed bounded convex set in a reflexive Banach space E, and $h: X \to \boldsymbol{R}$ a l. s. c. quasiconvex function. Then h attains its minimum on X.*

Finally, note that Mazur and Schauder applied Corollary 1 to a number of concrete problems in calculus of variations; these results were never published. See Granas [3].

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