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CONVEX SPACES AND KKM FAMILIES OF SUBSETS

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In [10], M.-H. Shih obtained a covering theorem for convex sets, and then applied to give a simple proof of a matching theorem of Ky Fan [5] concerning closed coverings of convex sets in a topological vector space (t.v.s.). In the present paper, we apply Shih's theorem to obtain a new type of the classical Knaster-Kuratowski-Mazurkiewicz theorem (simply, KKM theorem) and a new fixed point theorem including generalizations or variations of the Fan-Browder theorem due to a number of authors [1], [2], [6], [7], [11].

A *convex space* X [8] is a nonempty convex set X (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. In fact, we may regard that X has the relative finite topology. For a nonempty subset D of X, a family $\{Gx : x \in D\}$ of subsets of X is called a *KKM family* if $\mathrm{co}\{x : x \in A\} \subset \bigcup \{Gx : x \in A\}$ for any finite subset A of D, where co denotes the convex hull. $G : D \to 2^X$ is also called a *KKM multifunction.*

The following is due to Shih [10, Theorem 1]:

THEOREM 1. *Let A be a nonempty finite subset of a convex space X. If $\{Gx : x \in A\}$ is a KKM family of open subsets of X, then shere exists a KKM family $\{Fx : x \in A\}$ of closed subsets of X such that $Fx \subset Gx$ for each $x \in A$.*

We recall the following version of Ky Fan's generalization [4] of the KKM theorem due to Dugundji and Granas [3] and Lassonde [8]:

THEOREM 2. *Let D be a nonempty subset of a convex space X. If $\{Fx : x \in D\}$ is a KKM family of closed subsets of X, then it has the finite intersection property.*

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Combining Theorems 1 and 2, we obtain

THEOREM 3. *Let D be a nonempty subset of a convex space X. If $\{Gx : x \in D\}$ is a KKM family of open subsets of X, then it has the finite intersection property.*

Proof. Let A be a nonempty finite subset of D. Then by Theorem 1, there exists a KKM family $\{Fx : x \in A\}$ of closed subsets of X such that $Fx \subset Gx$ for each $x \in A$. By Theorem 2, $\cap \{Fx : x \in A\} \neq \phi$. Therefore, we have $\cap \{Gx : x \in A\} \neq \phi$.

REMARKS.

(3.1) If X is a convex subset of a Hausdorff t.v.s. E, then we may assume that X has the relative topology w.r.t. the finite topology of E. Therefore, in this case, Theorem 3 reduces to Kim [7, Theorem 1], [6, Theorem 2].

(3.2) Let $D = \{x_1, x_2, \dots, x_n\}$ be the set of vertices of an $(n-1)$-simplex in $X = \boldsymbol{R}^n$, and $G : D \to 2^X$ an open valued KKM multifunction. Then, by Theorem 3, we have $\cap_{i=1}^n Gx_i \neq \phi$. This is due to Kim [6, Theorem 1].

The following is a unified generalization of the Fan–Browder fixed point theorem and a result recently obtained by Kim [6, Theorem 3].

THEOREM 4. *Let X be a convex space and $S, T : X \to 2^X$ multifunctions satisfying*

(a) *$Tx \subset Sx$ for each $x \in X$,*

(b) *$S^{-1}y$ is convex for each $y \in X$, and*

(c) *$\{X \backslash Tx : x \in X\}$ is not a KKM family.*

Then S has a fixed point.

Proof. Let $Gx \equiv X \backslash Tx$ for each $x \in X$. Since $G : X \to 2^X$ is not a KKM multifunction, there exist a finite subset $\{x_1, x_2, \dots, x_k\}$ of X and an $x_0 \in \mathrm{co}\{x_1, x_2, \dots, x_k\}$ such that $x_0 \notin \cup_{i=1}^k Gx_i$. Therefore, $x_0 \in \cap_{i=1}^k Tx_i \subset \cap_{i=1}^k Sx_i$, and equivalently $x_i \in S^{-1}x_0$ for all i, $1 \leq i \leq k$. Since x_0 is a convex combination of x_i's and $S^{-1}x_0$ is convex, we have $x_0 \in S^{-1}x_0$, that is, $x_0 \in Sx_0$.

We have some consequences of Theorem 4.

THEOREM 5. *In Theorem* 4, *the condition* (c) *can be replaced by the following without affecting the conclusion*:

(c_1) *Tx is closed for each $x \in X$, and there exists a finite subset A of X such that $X = \bigcup \{Tx : x \in A\}$.*

Proof. Since $Gx \equiv X \backslash Tx$ is open for each $x \in X$ and $\bigcap \{Gx : x \in A\} = \phi$, by Theorem 3, $\{Gx : x \in X\}$ is not a KKM family. Hence, (c_1) implies (c).

REMARKS.

(5. 1) If X is a nonempty convex subset of a Hausdorff t. v. s. and $T \equiv S$, Theorem 5 reduces to Kim [6, Theorem 3] and [7, Theorem 4].

(5. 2) In [6], [7], Kim used Theorem 5 to obtain some applications.

THEOREM 6. *In Theorem* 4, *the condition* (c) *can be replaced by the following without affecting the conclusion*:

(c_2) *X is compact, Tx is open for each $x \in X$, and $T^{-1}y$ is nonempty for each $y \in X$.*

Proof. Since $T^{-1}y$ is nonempty for each $y \in X$ and X is compact, X is covered by a finite number of open Tx_i's, say, $1 \leq i \leq n$. Then we have $\bigcap_{i=1}^{n} (X \backslash Tx_i) = X \backslash \bigcup_{i=1}^{n} Tx_i = \phi$. Therefore, by Theorem 2, $\{X \backslash Tx : x \in X\}$ is not a KKM family, and hence, (c_2) implies (c).

REMARKS.

(6. 1) Theorem 6 is due to Ben-El-Mechaiekh, Deguire, and Granas [1, Theorem 1] and Simons [11, Theorem 4. 3]. In [1], the authors used Theorem 6 to obtain applications to variational inequalities of Hartman-Stampacchia and Browder and to the minimax inequality of Ky Fan.

(6. 2) For $S \equiv T$, Theorem 6 reduces to the Fan-Browder fixed point theorem [2].

(6. 3) In our previous work [9], we obtined far reaching generalizations of the Fan-Browder theorem and their applications in various fields.

For $S \equiv T$, Theorem 4 reduces to the following:

THEOREM 7. *Let X be a convex space and $T: X \to 2^X$ a multifunction such that $T^{-1}y$ is convex for each $y \in X$. Then T has a fixed point if and only if $\{X \setminus Tx : x \in X\}$ is not a KKM family.*

Proof. If $x_0 \in X$ is a fixed point of T, then $x_0 \notin X \setminus Tx_0$. Hence $\{X \setminus Tx : x \in X\}$ is not a KKM family. The converse follows from Theorem 4 with $T \equiv S$.

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