Nonlinear Analysis, Theory, Methods & Applications, Vol. 14, No. 2, pp. 159–165, 1990.
Printed in Great Britain.

0362-546X/90 $3.00 + .00

ON GENERALIZED ORDERING PRINCIPLES IN NONLINEAR ANALYSIS

BYUNG GAI KANG
Department of Mathematics, Sungshin Women's University, Seoul, 136-732, Korea

and

SEHIE PARK
Department of Mathematics, Seoul National University, Seoul, 151-742, Korea

(*Received* 10 *April* 1988; *received for publication* 20 *March* 1989)

Key words and phrases: Ordered sets, countably inductively ordered sets, d-asymptotic sequences, d-maximal elements, $\leq$-complete, u.s.c. maps.

1. INTRODUCTION

IN NONLINEAR analysis, a number of ordering principles have appeared. In fact, the well-known lemma of Bishop and Phelps [2] on the existence of maximal elements in certain partially ordered complete subsets of a normed linear space has been extended by a number of authors for various purposes, e.g. Phelps [9], Ekeland [6], Brøndsted [4], Brézis and Browder [3], Altman [1], Turinici [10, 11], Dancs, Hegedüs and Medvegyev [5], and Park [7, 8]. The most general form of such extensions known to us is Turinici's. Indeed, Turinici generalized Altman's generalization of the Brézis–Browder ordering principle which contains the results of Brøndsted [4] and Ekeland [6].

The aim of this paper is to establish theorems which generalize Turinici's ordering principle and to apply them directly to the proofs of the above-mentioned results. In certain cases, we obtain substantial generalizations of known results.

2. A GENERAL ORDERING PRINCIPLE

Let X be a nonempty set and $\leq$ a quasi-order (preorder or pseudo-order, that is, a reflexive and transitive relation) on X. Throughout this paper, $(X, \leq)$ will be called an *ordered set*. Let $S(x) = \{y \in X \mid x \leq y\}$ for $x \in X$, and $\leqq$ be the usual order in the extended real number system.

We begin with our basic results.

LEMMA. Let $(X, \leq)$ be an ordered set and $\{x_n\}$ a nondecreasing sequence in X with an upper bound $v \in X$, and $\{A_n\}$ a sequence of subsets of X such that $S(x_n) \subset A_n$ for each $n \in \mathbb{N}$. Then $S(v) \subset \bigcap_{n=1}^{\infty} A_n$.

Proof. Since $x_n \leq v$, we have $S(v) \subset S(x_n)$ for each $n \in \mathbb{N}$. ■

An ordered set $(X, \leq)$ is said to be *countably inductive* (a *CIO set*) if every nondecreasing sequence has an upper bound (cf. [6]).

THEOREM 1. Let $(X, \leq)$ be a CIO set and $\{A_n\}$ a sequence of subsets of X such that

$$\text{for any } x \in X \text{ and } n \in \mathbb{N}, \text{ there is a } y \in S(x) \text{ satisfying } S(y) \subset A_n. \tag{1.1}$$

Then for any $x \in X$, there is a $v \in S(x)$ such that $S(v) \subset \bigcap_{n=1}^{\infty} A_n$.

Proof. Let $x \in X$. By (1.1), we can choose $x_1 \in S(x)$ such that $S(x_1) \subset A_1$. Suppose that $x \leq x_1 \leq \cdots \leq x_n$ are chosen so that $S(x_i) \subset A_i$, $i = 1, 2, \ldots, n$. Again by (1.1), there is an $x_{n+1} \in S(x_n)$ such that $S(x_{n+1}) \subset A_{n+1}$. By induction, we obtain a nondecreasing sequence $\{x_n\}$ such that $S(x_n) \subset A_n$ for each $n \in \mathbb{N}$. Let v be an upper bound for $\{x_n\}$. The conclusion follows from the lemma. ■

THEOREM 2. Let $(X, \leq)$ be a CIO set and $d: X \times X \to [0, \infty]$ a function satisfying

$$\text{for any } x \in X \text{ and } \varepsilon > 0, \text{ there is a } y \in S(x) \text{ such that } d(y_1, y_2) < \varepsilon \text{ if } y \leq y_1 \leq y_2. \tag{2.1}$$

Then for any $x \in X$, there is a $v \in S(x)$ such that $d(v, z) = 0$ for all $z \in S(v)$.

Proof. Let $A_n = \{y \in X \mid d(y_1, y_2) < 1/n \text{ if } y \leq y_1 \leq y_2\}$ for $n \in \mathbb{N}$. Since (2.1) implies (1.1), the conclusion follows from theorem 1. ■

COROLLARY. Under the assumptions of theorem 2, if $d(x, y) = 0$ implies $x = y$, then every selfmap $f: X \to X$ satisfying $x \leq fx$ for each $x \in X$ has a fixed point.

In [11], a function $d: X \times X \to [0, \infty]$ is called a *$\leq$-quasi metric* if $d(x, x) = 0$ for all $x \in X$ and the following holds:

$$\text{for every } \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that } x \leq y \leq z \text{ and } d(x, y), d(x, z) < \delta \text{ imply } d(y, z) < \varepsilon. \tag{*}$$

For any function $d: X \times X \to [0, \infty]$, a sequence $\{x_n\}$ in X is said to be *d-asymptotic* if $\liminf_{n \to \infty} d(x_n, x_{n+1}) = 0$, and an element $v \in X$ is said to be *d-maximal* if $d(v, z) = 0$ for all $z \in S(v)$.

Theorem 2 gives a sufficient condition for the existence of d-maximal element of a CIO set.

THEOREM 3. Let $(X, \leq)$ be a CIO set and $d: X \times X \to [0, \infty]$ a $\leq$-quasi metric on X satisfying

$$\text{for any } x \in X \text{ and } \varepsilon > 0, \text{ there is a } y \in S(x) \text{ such that } d(y, z) < \varepsilon \text{ for all } z \in S(y). \tag{3.1}$$

Then for any $x \in X$, there is a d-maximal element $v \in S(x)$.

Proof. In view of theorem 2, it suffices to show that (*) and (3.1) imply (2.1). Let $x \in X$ and $\varepsilon > 0$ be given. Choose $\delta > 0$ so that (*) holds. By (3.1), there is a $y \in S(x)$ such that $d(y, z) < \delta$ for all $z \in S(y)$. So if $y \leq y_1 \leq y_2$, $d(y, y_1) < \delta$ and $d(y, y_2) < \delta$. Hence by (*), $d(y_1, y_2) < \varepsilon$. ■

Consider the following condition:

$$\text{any nondecreasing sequence in } X \text{ is } d\text{-asymptotic.} \qquad (**)$$

It is easy to show that (**) implies (3.1). So theorem 3 generalizes Turinici's main result [11]. Note also that if d is a metric, then (2.1) and (3.1) are equivalent to the following:

$$\begin{array}{l}\text{for any } x \in X \text{ and } \varepsilon > 0, \text{ there is a } y \in S(x)\\ \text{such that } \operatorname{diam} S(y) < \varepsilon.\end{array} \qquad (2.1)'$$

Now, Altman's principle [1] can be improved as follows.

THEOREM 4. Let $(X, \leq)$ be a CIO set and $d: X \times X \to \mathbb{R} \cup \{\infty\}$ a function satisfying

$$\begin{array}{l}\text{there is a function } c: X \to \mathbb{R} \text{ such that}\\ c(x) \leqq d(x, y) \leqq 0 \text{ for all } x \in X \text{ and } y \in S(x),\end{array} \qquad (4.1)$$

and

$$\begin{array}{l}\text{for any } x \in X \text{ and } \varepsilon > 0, \text{ there is a } y \in S(x)\\ \text{such that } -\varepsilon \leqq c(z) \text{ for all } z \in S(y).\end{array} \qquad (4.2)$$

Then for all $x \in X$, there is a d-maximal element $v \in S(x)$.

Proof. Let $A_n = \{y \in X \mid -1/n \leqq c(z) \text{ for all } z \in S(y)\}$ for $n \in \mathbb{N}$. Since (4.2) implies (1.1), in view of theorem 1, for any $x \in X$, there is a $v \in S(x)$ such that $S(v) \subset \bigcap_{n=1}^{\infty} A_n$. Thus $0 \leqq c(z)$ for all $z \in S(v)$. By (4.1), $0 \leqq c(v) \leqq d(v, z) \leqq 0$ for all $z \in S(v)$. ■

As a direct application of theorem 1, we obtain the following generalization of the Brézis-Browder principle [3].

THEOREM 5. Let $(X, \leq)$ be a CIO set and $\phi: X \to \mathbb{R} \cup \{\infty\}$ a nonincreasing function $\not\equiv \infty$, bounded from below. Then either
(1) ϕ attains its infimum on X, or
(2) there is an $x \in X$ such that $\inf_{S(x)} \phi > \inf_X \phi$.

Proof. Suppose that $\inf_{S(x)} \phi = \inf_X \phi$ for all $x \in X$. For $n \in \mathbb{N}$, let $A_n = \{x \in X \mid \phi(x) < \inf_X \phi + 1/n\}$. We claim that (1.1) holds. Let $x \in X$ and $n \in \mathbb{N}$. Since $\inf_{S(x)} \phi = \inf_X \phi$, there is a $y \in S(x)$ such that $\phi(y) < \inf_X \phi + 1/n$. Since ϕ is nonincreasing, $z \in S(y)$ implies $\phi(z) \leqq \phi(y) < \inf_X \phi + 1/n$, that is, $z \in A_n$. Therefore, (1.1) holds. By theorem 1, there is a $v \in S(x)$ such that $s(v) \subset \bigcap_{n=1}^{\infty} A_n$. Thus $\phi(v) = \inf_X \phi$. ■

The following useful formulation of the Brézis-Browder principle is due to Ekeland [6].

THEOREM 6 [6]. Let $(X, \leq)$ be a CIO set and $\phi: X \to \mathbb{R} \cup \{\infty\}$ a nonincreasing function bounded from below. Then for any $x \in X$, there is a $y \in S(x)$ such that $\phi(y) = \phi(z)$ for all $z \in S(y)$.

Proof. Restrict ϕ to the set $S(x)$. If $\phi \equiv \infty$ on $S(x)$, the conclusion is clear. Suppose that $\phi \not\equiv \infty$ on $S(x)$. Since ϕ is nonincreasing, we have $\inf_{S(x)} \phi = \inf_{S(y)} \phi$ for all $y \in S(x)$. Therefore, by theorem 5, ϕ attains its infimum on $S(x)$. Let $\phi(y) = \inf_{S(x)} \phi$. Then $y \leq z$ implies $\phi(y) = \phi(z)$. ■

Note that we can also deduce theorem 6 from theorem 4 by letting $d(x, y) = \phi(y) - \phi(x)$.

In [3], Brézis and Browder applied their principle to a number of diverse results in nonlinear functional analysis, and also noted that the following result of Brøndsted [4] is a consequence of their principle. For the completeness, we give its proof by using theorem 6. Note also that Brøndsted's original proof used Zorn's lemma.

THEOREM 7 [4]. Let $(X, \leq)$ be an ordered set, $\mathcal{U}$ a Hausdorff uniformity on X and $\phi: X \to \mathbb{R} \cup \{\infty\}$ a nonincreasing function bounded from below. Suppose that

$$S(x) \text{ is complete for each } x \in X, \tag{7.1}$$

and

$$\begin{aligned} &\text{for each } U \in \mathcal{U}, \text{ there is a } \delta > 0 \text{ such that} \\ &x \leq y \text{ and } \phi(x) < \phi(y) + \delta \text{ imply } (x, y) \in U. \end{aligned} \tag{7.2}$$

Then for any $x \in X$ with $\phi(x) < \infty$, there exists a maximal element $v \in S(x)$.

Proof. We may assume that $\phi \not\equiv \infty$. Let $D = \{x \in X \mid \phi(x) < \infty\}$. We claim that $(D, \leq)$ is a CIO set. Let $\{x_n\}$ be a nondecreasing sequence in D. Since ϕ is nonincreasing, $S(x_n) \subset S(x_{n-1}) \subset D$ for all n. We will show that $\{x_n\}$ is a Cauchy sequence in $S(x_1)$. Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $V \circ V^{-1} \subset U$. By (7.2), there exists a $\delta > 0$ such that $x \leq y$ and $\phi(x) < \phi(y) + \delta$ implies that $(x, y) \in V$. Let $a = \inf_n \phi(x_n)$. Choose N so that $1/N < \delta$ and $n \geqq N$ implies $a \leqq \phi(x_n) \leqq a + 1/N$. Then $n \geqq m \geqq N$ implies that

$$\phi(x_m) - \phi(x_n) \leqq \phi(x_N) - a < 1/N < \delta.$$

Hence $(x_m, x_n) \in V$. Since $V \circ V^{-1} \subset U$, $(x_m, x_n) \in U$ and $(x_n, x_m) \in U$. Therefore $\{x_n\}$ is a Cauchy sequence in $S(x_1)$. Thus by (7.1), $\{x_n\}$ converges to some $v \in S(x_1)$. Obviously, this v is an upper bound for $\{x_n\}$. By theorem 6, for any $x \in D$, there exists a $y \in S(x)$ such that $y \leq z$ implies $\phi(y) = \phi(z)$. Now we claim that y is a maximal element. By (7.2), $(y, z) \in U$ for all $z \in S(y)$ and $U \in \mathcal{U}$. Since $\mathcal{U}$ is a Hausdorff uniformity, $y = z$ for all $z \in S(y)$. This completes the proof. ∎

As was shown in [4], theorem 7 generalizes the well-known variational principles of Ekeland [6] and the celebrated 1961 lemma of Bishop and Phelps [2].

3. METRIC COMPLETENESS AND ORDERING PRINCIPLES

If X is a metric space with a quasi-order $\leq$, then the countable inductiveness condition in the previous section can be replaced by, in some sense, completeness of X.

In the frame of theorem 7, we say that $\leq$ is *closed* if $S(x)$ is closed for all $x \in X$, and that X is $\leq$-*complete* if every nondecreasing Cauchy sequence in X converges [10]. Note that if X is complete, then it is $\leq$-complete for any quasi-order $\leq$ on X. Note also that the completeness in (7.1) can be replaced by $\leq$-completeness without affecting the conclusion. Moreover, in theorem 2, if (X, d) is a $\leq$-complete metric space, then X is not necessarily CIO. Precisely, we have the following equivalent results.

THEOREM 8. Let (X, d) be a metric space with a quasi-order $\leq$. If (2.1) holds, then for any $x \in X$ such that $S(x)$ is complete, there is a maximal element $v \in S(x)$.

THEOREM 8′. Let (X, d) be a metric space with a quasi-order $\leq$. If X is $\leq$-complete and (2.1) holds, then for any $x \in X$, there exists a maximal element $v \in S(x)$.

Proof. Since theorem 8 and theorem 8′ are equivalent, we prove the latter. Let $x \in X$. By (2.1), there is an $x_1 \in S(x)$ such that $d(x_1, z) < \frac{1}{2}$ for all $z \in S(x_1)$. For $y, z \in S(x_1)$, we have $d(y, z) \leqq d(x_1, y) + d(x_1, z) < 1$. Thus $\operatorname{diam} S(x_1) \leqq 1$. Inductively, we can select a nondecreasing sequence $\{x_n\}$ in X such that $\operatorname{diam} S(x_n) \leqq 1/n$. Then $\{x_n\}$ is a Cauchy sequence and converges to some $v \in X$. Since each $S(x_n)$ is closed and $v = \lim_{m \geqq n} x_m$, $v \in S(x_n)$. If $v \leq z$, then $z \in \bigcap_{n=1}^{\infty} S(x_n)$. Since $\operatorname{diam} S(x_n) \to 0$, $z = v$. ■

Note that theorem 8′ improves theorem 3.1 in [10]. As an application of theorem 8, we prove the following fixed point theorem for a multi-valued function.

THEOREM 9. Let (X, d) be a metric space and $T: X \to 2^X \setminus \{\varnothing\}$ a map satisfying

$$\overline{T(x_0)} \text{ is complete for some } x_0 \in X, \tag{9.1}$$

$$\text{for any } x, y \in \overline{T(x_0)},\ y \in \overline{T(x)} \text{ implies } T(y) \subset \overline{T(x)}, \tag{9.2}$$

and

$$\text{for any } x \in \overline{T(x_0)} \text{ and } \varepsilon > 0, \text{ there is a } y \in \overline{T(x)} \text{ such that } \operatorname{diam} T(y) < \varepsilon. \tag{9.3}$$

Then for any $x \in \overline{T(x_0)}$, there exists a stationary point $y \in \overline{T(x)}$, that is, $T(y) = \{y\}$.

Proof. Define $\leq$ on $\overline{T(x_0)}$ by $x \leq y$ iff $x = y$ or $y \in \overline{T(x)}$. Then $\leq$ is a closed quasi-order on $\overline{T(x_0)}$ by (9.2). Since (9.3) implies (2.1), by theorem 8′, for any $x \in \overline{T(x_0)}$, there is a maximal element $y \in S(x)$. If $z \in T(y)$, then $z \geq y$ implies $z = y$. Hence $T(y) = \{y\}$. ■

Note that theorems 3.1 and 3.2 in [5] follow from theorems 9 and 8′, respectively.

THEOREM 10. Let X be a set, $d: X \times X \to [0, \infty]$, and $\phi: X \to \mathbb{R}$ a function bounded from above. Define an order $\leq$ on X by

$$x \leq y \text{ iff } d(x, y) \leqq \phi(y) - \phi(x).$$

If X is a CIO set with respect to $\leq$, then for any $x \in X$, there is a d-maximal element $v \in S(x)$.

Proof. In view of theorem 3, it suffices to show that (**) holds. Let $\{x_n\}$ be a nondecreasing sequence in X and $a = \sup \phi(x_n) < \infty$. For $m \geqq n$, we have $x_m \geq x_n$ and

$$d(x_m, x_n) \leqq \phi(x_m) - \phi(x_n).$$

Since $\phi(x_n) \uparrow a$, we have $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Thus (**) holds. Note also that $\{x_n\}$ is a Cauchy sequence. ■

From each of theorems 2, 8, 9 and 10, we obtain the following extension of the lemma in [2] due to Phelps [9].

THEOREM 11. Let (X, d) be a complete metric space and $\phi: X \to \{-\infty\} \cup \mathbb{R}$ an u.s.c. function bounded from above on $X_1 = \{x \in X \mid \phi(x) > -\infty\}$. Define an order $\le$ on X_1 as in theorem 10. Then for any $x \in X_1$, there exists a maximal element $v \in S(x)$.

Proof. In view of theorem 10, it suffices to show that $(X_1, \le)$ is a CIO set. Let $\{x_n\}$ be a nondecreasing sequence in X_1 and $a = \sup \phi(x_n) < \infty$. Since $\{x_n\}$ is Cauchy, it converges to some $y \in X$. Since ϕ is u.s.c., we have $\phi(y) \geqq \limsup \phi(x_n) = a$ and hence $y \in X_1$. And since ϕ is u.s.c., each $S(x_n)$ is closed. But then $y = \lim_{m \geqq n} x_m \in S(x_n)$ for each n. This completes the proof. ■

Finally, we obtain the following.

THEOREM 12. Let X be a complete metric space and $\phi: X \to \{-\infty\} \cup \mathbb{R}$ an u.s.c. function, $\not\equiv -\infty$, bounded from above. Let $\varepsilon > 0$ be given and a point $u \in X$ satisfy $\phi(u) < \sup_X \phi - \varepsilon$, and $A = \{x \in X \mid \varepsilon d(x, u) \leqq \phi(x) - \phi(u)\}$.

Then the following equivalent statements hold:

(i) There exists a point $v \in A$ such that

$$\forall\, w \in X\backslash\{v\}, \qquad \varepsilon d(v, w) > \phi(w) - \phi(v).$$

(ii) If $T: A \to 2^X$ satisfies the condition:

$$\forall\, x \in A\backslash T(x) \qquad \exists\, y \in X\backslash\{x\}, \qquad \varepsilon d(x, y) \leqq \phi(y) - \phi(x),$$

then T has a fixed point $v \in A$, that is, $v \in T(v)$.

(iii) If $f: A \to X$ is a map satisfying

$$\varepsilon d(x, fx) \leqq \phi(fx) - \phi(x)$$

for $x \in A$, then f has a fixed point.

(iv) If $T: A \to 2^X\backslash\{\phi\}$ satisfies the condition:

$$\forall\, x \in A \qquad \forall\, y \in T(x), \qquad \varepsilon d(x, y) \leqq \phi(y) - \phi(x),$$

then T has a stationary point $v \in A$, that is, $T(v) = \{v\}$.

Proof. Since the equivalency of (i)-(iv) is a consequence of results in [7, 8], we prove only (iii) from theorem 11. Note that A is closed since ϕ is u.s.c. and that $fA \subset A$. Therefore, f is a selfmap of the complete metric space A. Define an order $\le$ on (A, d) as in theorem 10. Then A has a maximal element v, which should be fixed under f as in corollary to theorem 2. ■

Note that $A \subset \{x \in X \mid \phi(x) \geqq \phi(u), d(x, u) \leqq 1\} \subset \bar{B}(u, 1) \cap X$. Theorem 12 (i) is the dual of Ekeland's variational principle [6], and the dual of (iii) is known as the Caristi-Kirk-Browder fixed point theorem.

Actually, theorem 11 and theorem 12 are equivalent. For this purpose, we obtain theorem 11 from theorem 12 (i). For any $x \in X_1$, choose an $\varepsilon > 0$ such that $\phi(x) \geqq \sup_X \phi - \varepsilon$, and replace ϕ by ϕ/ε. Let $A = \{z \in X_1 \mid z \ge x\}$. Then by theorem 12 (i), there exists a $v \in A$ such that $d(v, w) > \phi(w) - \phi(v)$ for any $w \neq v$. Then $v \ge x$ and v is maximal. For, if $v' \ge v$, then $d(v', v) \leqq \phi(v') - \phi(v)$, which is absurd.

As in theorem 10, certain assumption in theorem 12 is not essential.

REFERENCES

1. ALTMAN M., A generalization of the Brézis-Browder principle on ordered sets, *Nonlinear Analysis* **6**, 157–165 (1982).
2. BISHOP E. & PHELPS R. R., The support functionals of a convex set, *Proc. Symp. Pure Math. VII. Convexity, Am. Math. Soc.* 27-36 (1963).
3. BRÉZIS H. & BROWDER F. E., A general principle on ordered sets in nonlinear functional analysis, *Adv. Math.* **21**, 355-364 (1976).
4. BRØNDSTED A., On a lemma of Bishop and Phelps, *Pacific J. Math.* **55**, 335-341 (1974).
5. DANCS S., HEGEDÜS M. & MEDVEGYEV P., A general ordering and fixed point principle in complete metric spaces, *Acta Sci. Math.* **46**, 381-388 (1983).
6. EKELAND I., Nonconvex minimization problems, *Bull. Am. Math. Soc.* **1**, 443–474 (1979).
7. PARK S., Equivalent formulations of Ekeland's variational principle and their applications, *MSRI-Korea Publ.* **1**, 55-68 (1986).
8. PARK S., Some applications of Ekeland's variational principle to fixed point theory, in *Approximation Theory and Applications* (Edited by S. P. SINGH), *Res. Notes Math.* **133**, 159-172 (1985).
9. PHELPS R. R., Weak* support points of convex sets in E*, *Israel J. Math.* **2**, 177-182 (1964).
10. TURINICI M., Maximal elements in a class of order complete metric spaces, *Math. Japonica* **25**, 511-517 (1980).
11. TURINICI M., A generalization of Altman's ordering principle, *Proc. Am. Math.* **90**, 128-132 (1984).