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NUMER. FUNCT. ANAL. AND OPTIMIZ., 11(1&2), 101-110 (1990)

GENERALIZED MATCHING THEOREMS FOR CLOSED COVERINGS OF CONVEX SETS

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ABSTRACT. In this paper we use fixed point and coincidence theorems due to Park [8] to give matching theorems concerning closed coverings of nonempty convex sets in a real topological vector space. Our new results extend previously given ones due to Ky Fan [2], [3], Shih [10], Shih and Tan [11], and Park [7].

1. Introduction

In [2], Ky Fan obtained a matching theorem for two closed coverings of a compact convex set, and used it to obtain a further generalization of Shapley's generalization [9] of the classical KKM theorem. Note that Shapley's theorem is a useful result in game theory. Further, in [3], Ky Fan extended his result to a matching theorem for two closed coverings of a paracompact convex set, and applied it to give a new proof of the Brouwer fixed point theorem and a new generalization of Shapley's theorem. Recently, Ky Fan's matching theorems in [2], [3] are extended by Shih [10], Shih and Tan [11], and Park [1].

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1980 *Mathematics Subject Classification.* Primary 52A07; Secondary 47H10.
Key words and phrases. KKM theorem, upper hemicontinuous multifunction, weakly inward (outward) set, fixed point, coincidence point, paracompact convex set, locally finite family, matching theorem.
Supported in part by the Korea Science and Engineering Foundation, 1988.

The aim in this paper is to generalize and unify all those matching theorems in [2], [3], [7], [10], and [11] without assuming the paracompactness.

Theorem 0, the main tool in this paper, is a fixed point theorem recently due to the author [8] and a generalization of the main result of Jiang [4].

Theorems 1 and 2 are equivalent formulations of Theorem 0. Theorem 1 extends Park [7, Theorem 7], Ky Fan [3, Theorem 10], and Lee and Tan [5, Theorem 6], and Theorem 2 extends Park [7, Corollary 7.1] and Aubin and Ekeland [1, Theorem 6.4.11].

Theorem 3 is a matching theorem for two closed coverings of a convex set, and extends Ky Fan [3, Theorem 11], [2, Theorem 1], and Park [7, Theorem 9].

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Theorem 4 is a general covering theorem of a convex set, and generalizes Shih [10, Theorem 2] and Shih and Tan [11, Theorems 1 and 2]. Theorem 5 is a particular case of Theorem 4 including Ky Fan [3, Theorem 12], [2, Theorem 2], and Park [7, Theorem 10].

2. Preliminaries

[7],

We keep the terminology and notations in [8].

Let E be a real Hausdorff topological vector space (t.v.s.) and E^* its dual space equipped with the topology of compact convergence. For $\phi \in E^*$ and $U, V \subset E$, we denote $d_\phi(U, V) = \inf\{|\phi(u-v)| : u \in U, v \in V\}$. Let $cc(E)$ denote the class of all nonempty closed convex subsets of E, and $kc(E)$ the subclass of all compact sets in $cc(E)$. Bd, $-$, and co denote the boundary, closure, and convex hull, resp.

A nonempty subset L of a convex set X in E is called a *c–compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

Let Y be a topological space. A multifunction $F : Y \to 2^E$ is said to be *upper hemicontinuous* (u.h.c.) if for each $\phi \in E^*$ and for any real a, the set $\{y \in Y : \sup \phi(Fy) < a\}$ is open in Y. See [1] and [11]. Any upper semicontinuous (u.s.c.) multifunction is u.h.c.

Let $X \subset E$ and $x \in E$. The *inward* and *outward* sets of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

Their closures are called *weakly inward* (resp. *outward) sets.*

The following is known recently:

THEOREM 0 [8] ~~THEOREM 6~~]. *Let X be a nonempty convex set in a real t.v.s. E, L a c–compact subset of X, K a nonempty compact subset of X, and $F : X \to 2^E$. Suppose that, for any $\phi \in E^*$, $\{x \in X : \phi x \geq \inf \phi(Fx)\}$ is closed, and that either*

(A) E^* separates points of E and $F : X \to kc(E)$, or

(B) E is locally convex and $F : X \to cc(E)$.

(1) If $d_\phi(Fx, \overline{I}_X(x)) = 0$ for every $x \in K \cap BdX$ and $\phi \in E^*$, and $d_\phi(Fx, \overline{I}_L(x)) = 0$ for every $x \in X \backslash K$ and $\phi \in E^*$, then F has a fixed point.

(2) If $d_\phi(Fx, \overline{O}_X(x)) = 0$ for every $x \in K \cap BdX$ and $\phi \in E^*$, and $d_\phi(Fx, \overline{O}_L(x)) = 0$ for every $x \in X \backslash K$ and $\phi \in E^*$, then F has a fixed point.

3. Main results

First we give some equivalent forms of Theorem 0.

THEOREM 1. *Let X, L, and K be as in Theorem 0. Let $F, G : X \to 2^E$ be functions and $S = I + F - G$, where $I : X \to E$ is the inclusion. Suppose that, for any $\phi \in E^*$, $\{x \in X : \phi x \geq \inf \phi(Sx)\}$ is closed and that either*

(A) E^* separates points of E and $F, G : X \to kc(E)$, or

(B) E is locally convex, $F, G : X \to cc(E)$, and Fx or Gx is compact for each $x \in X$.

If $d_\phi(Sx, \overline{I}_X(x)) = 0$ for every $x \in K \cap BdX$ and $\phi \in E^*$, and $d_\phi(Sx, \overline{I}_L(x))$ $= 0$ for every $x \in X \backslash K$ and $\phi \in E^*$, then F and G have a coincidence point.

Proof: In case (A), we note that $S : X \to kc(E)$ and, in case (B), $S : X \to cc(E)$. Therefore, by Theorem 0(1), S has a fixed point, or equivalently, F and G have a coincidence point.

REMARKS: (1) Note that if F and G are u.h.c., so is S, and hence S satisfies the hypothesis of Theorem 1. Therefore, [7, Theorem 7] follows from Theorem 1

(2) By exchanging the roles of F and G, it is clear that Theorem 1 remains valid if the weakly inward sets are replaced by the corresponding weakly outward sets.

note that

(3) ~~Since~~ $Sx \cap \overline{I}_X(x) \neq \emptyset$ implies $d_\phi(Sx, \overline{I}_X(x)) = 0$. ~~[7, Theorem 7] follows from Theorem 1~~. Particular forms of the former, mainly for case (B), are due to Ky Fan [3, Theorem 10], and Lee and Tan [5, Theorem 6].

(4) Note that Theorem 0 and Theorem 1 are equivalent. For, if $G = I$ or $F = I$, resp., Theorem 1 reduces to Theorem 0(1) or (2), resp. Moreover, if $G = 0$, we have the following

THEOREM 2. *Let X, L, and K be as in Theorem 1. Let $F : X \to 2^E$. Suppose that, for any $\phi \in E^*$, $\{x \in X : \inf \phi(Fx) \leq 0\}$ is closed and that either*

(A) E^* separates points of E and $F : X \to kc(E)$, or

(B) E is locally convex and $F : X \to cc(E)$.

If $d_\phi((I + F)x, \overline{I}_X(x)) = 0$ for every $x \in K \cap Bd\,X$ and $\phi \in E^*$, and $d_\phi((I + F)x, \overline{I}_L(x)) = 0$ for every $x \in X \backslash K$ and $\phi \in E^*$, then

(a) $\exists\, \overline{x} \in X$ such that $0 \in F\overline{x}$, and

(b) if F is u.h.c., then $\forall\, y \in X\ \exists\, \hat{x} \in X$ such that $y \in \hat{x} - F\hat{x}$.

Proof: (a) Note that Theorem 2(a) is equivalent to Theorem 0(1).

(b) Since the multifunction G defined by $Gx = Fx + y - x$ satisfies all the requirements for F in (a), G has a zero $\hat{x}$. This shows (b).

REMARKS: Theorem 2 contains [7, Corollary 7.1] and Aubin and Ekeland

[1, Theorem 6.4.11]. In [1] some applications of the later can be seen.

Using Theorem 1, we now prove a matching theorem for two closed coverings of a convex set.

THEOREM 3. *Let X be a nonempty convex set in a real locally convex t.v.s. E, L a c–compact subset of X, and K a nonempty compact subset of X. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two locally finite families of relatively closed subsets of X such that*

$$(a) \qquad \bigcup_{i\in I} A_i = \bigcup_{j\in J} B_j = X.$$

Let $\{C_i\}_{i\in I}$ and $\{D_j\}_{j\in J}$ be two corresponding families of nonempty subsets of E such that any finite union of the $C_i's$ is contained in a compact convex subset of E. Suppose that for each $x \in (K \cap Bd\,X) \cup (X\backslash K)$, there exists $i \in I$ and $j \in J$ such that $x \in A_i \cap B_j$, and

$$(b) \qquad d_\phi(x + C_i - D_j, \quad \overline{I}_X(x)) = 0$$

for every $x \in K \cap Bd\,X$ and $\phi \in E^*$, and

$$(c) \qquad d_\phi(x + C_i - D_j, \quad \overline{I}_L(x)) = 0$$

for every $x \in X\backslash K$. Then there exist two nonempty finite sets $I_0 \subset I$ and $J_0 \subset J$ such that

$$(\bigcap_{i\in I_0} A_i) \cap (\bigcap_{j\in J_0} B_j) \neq \emptyset$$

and

$$(\overline{\mathrm{co}} \bigcup_{i\in I_0} C_i) \cap (\overline{\mathrm{co}} \bigcup_{j\in J_0} D_j) \neq \emptyset.$$

Proof: For each $x \in X$, let

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$$I(x) = \{i : x \in A_i\}, \quad J(x) = \{J : x \in B_j\}$$

and let

$$Fx = \overline{\text{co}} \bigcup_{i\in I(x)} C_i, \quad Gx = \overline{\text{co}} \bigcup_{j\in J(x)} D_j.$$

As the families $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ are locally finite, $I(x)$ and $J(x)$ are finite. Therefore, the closed convex hull Fx of the finite union $\bigcup_{i\in I(x)} C_i$ is compact by hypothesis. Again by local finiteness of $\{A_i\}_{i\in I}$ for each $x \in X$ the complement $U(x)$ of $\bigcup_{i\notin I(x)} A_i$ in X is an open neighborhood of x in X. If $y \in U(x)$, then $I(y) \subset I(x)$ and therefore $Fy \subset Fx$. This implies that $F : X \to kc(E)$ is u.s.c. on X. Similarly, since $\{B_j\}_{j\in J}$ is locally finite, $G : X \to cc(E)$ is u.s.c. on X.

Let $S = I + F - G$. For $x \in (K \cap \operatorname{Bd} X) \cup (X\backslash K)$, we can find $i \in I(x)$ and $j \in J(x)$ satisfying (b) and (c). Therefore, $d_\phi(Sx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \operatorname{Bd} X$ and $\phi \in E^*$, and $d_\phi(Sx, \overline{I}_L(x)) = 0$ for every $x \in X\backslash K$ and $\phi \in E^*$. Now by Theorem 1, there is a point $\hat{x} \in X$ such that $F\hat{x} \cap G\hat{x} \neq \emptyset$. If we take $I_0 = I(\hat{x})$ and $J_0 = J(\hat{x})$, then the conclusion of Theorem 3 is verified.

REMARKS: (1) Furthermore, if any finite union of the D_j's is also contained in a compact convex subset of E, then $G : X \to kc(E)$ and Theorem 3 holds for a real t.v.s. E on which E^* separates points.

(2) The weakly inward sets in (b) and (c) can be replaced corresponding weakly outward sets in Theorem 3 if in the hypothesis "any finite union of the C_i's is contained in a compact convex subset of E", "the C_i's" is replaced by "the D_j's".

(3) In view of our previous work [8], Theorem 3 properly generalizes [7, Theorem 9].

(4) The conditions (b) and (c) are implied by the following:

$$C_i - D_j \text{ meets } \begin{cases} \bigcup_{\lambda>0} \lambda(X - x) & \text{if } x \in K \cap \operatorname{Bd} X; \\ \bigcup_{\lambda>0} \lambda(L - x) & \text{if } x \in X\backslash K. \end{cases}$$

Under this condition instead of (b) and (c), and under the assumption that $K = L$ is a nonempty compact convex subset of X, Theorem 3 reduces to

Ky Fan [3, Theorem 11]. For the special case of compact $X = K = L$ and finite sets I and J, [3, Theorem 11] is given as [2, Theorem 1].

The following is a general covering theorem of convex sets in a real locally convex t.v.s.

THEOREM 4. *Let X be a nonempty convex set in a real locally convex t.v.s. E, L a c–compact subset of X, and K a nonempty compact subset of X. Let $\{A_i\}_{i\in I}$ be a locally finite family of relatively closed subsets of X such that $\bigcup_{i\in I} A_i = X$, and let $\{C_i\}_{i\in I}$ be a family of nonempty subsets of E. Let $T : X \to 2^E$ be an u.h.c. map such that each Tx is a nonempty weakly compact convex set. If*

$$(d) \qquad d_\phi(\overline{co} \bigcup_{x\in A_i} (C_i + Tx),\ \overline{I}_X(x)) = 0$$

for evey $x \in K \cap Bd\,X$ and $\phi \in E^*$, and

$$(e) \qquad d_\phi(\overline{co} \bigcup_{x\in A_i} (C_i + Tx),\ \overline{I}_L(x)) = 0$$

for every $x \in X\backslash K$ and $\phi \in E^*$, then there exist a nonempty finite subset I_0 of I and a point $\hat{x} \in X$ such that

$$\hat{x} \in (\overline{co} \bigcup_{i\in I_0} (C_i + T\hat{x})) \cap (\bigcup_{i\in I_0} A_i).$$

Proof: For each $x \in X$, let $I(x) = \{i : x \in A_i\}$. Since $\bigcup_{i\in I} A_i = X$ and $\{A_i\}_{i\in I}$ is locally finite, each $I(x)$ is nonempty and finite. For each $x \in X$, let

$$Fx = \overline{co} \bigcup_{i\in I(x)} (C_i + Tx) \in cc(E)$$

and

$$Gx = \overline{co} \bigcup_{i \in I(x)} C_i.$$

Since for each $x \in X$,

$$\bigcup_{i \in I(x)} (C_i + Tx) = \bigcup_{i \in I(x)} C_i + Tx$$

and each Tx is weakly compact convex, we see that $F = G + T$. Again by local finiteness of $\{A_i\}_{i \in I}$, for each $x \in X$ the complement $U(x)$ of $\bigcup_{i \notin I(x)} A_i$ in X is an open neighborhood of x in X. If $y \in U(x)$, then $I(y) \subset I(x)$ and therefore $Gy \subset Gx$. Consequently, G is u.s.c. on X, and hence $F = G + T$ is u.h.c. By hypothesis, $d_\phi(Fx, \overline{I}_X(x)) = 0$ for each $x \in K \cap \operatorname{Bd} X$ and $\phi \in E^*$, and $d_\phi(Fx, \overline{I}_L(x)) = 0$ for each $x \in X \backslash K$ and $\phi \subset E^*$. By applying Theorem 0(B), there exists a point $\hat{x} \in X$ such that $\hat{x} \in F\hat{x}$. If we take $I_0 = I(\hat{x})$, then the conclusion of Theorem 4 is verified.

REMARKS: (1) Theorem 4 remains true if we replace the weakly inward sets in (d) and (e) by the corresponding weakly outward sets.

(2) The conditions (d) and (e) are implied by the following:

$$\overline{co} \bigcup_{x \in A_i} (C_i + Tx) \text{ meets } \begin{cases} \overline{I}_X(x) & \text{if } x \in K \cap \operatorname{Bd} X; \\ \overline{I}_L(x) & \text{if } x \in X \backslash K. \end{cases}$$

Under this condition instead of (d) and (e), Theorem 4 reduces to Shih and Tan [11, Theorem 1].

(3) In case where $X = K = L$ is compact and I is finite, Theorem 4 generalizes Shih and Tan [11, Theorem 2], and Shih [10, Theorem 2].

(4) Another Ky Fan matching theorem [3, Theorem 2] can be compared with a special case, when $Tx = 0$ for each $x \in X = K = L$ and each C_i is a singleton in X, of Theorem 4. Note that [3, Theorem 2] is generalized in other direction by the author [6, Theorem 7] with some applications.

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The following is a consequence of Theorems 3 and 4.

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Received: July 5, 1989
Revised: January 19, 1990
Accepted: April 19, 1990