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FIXED POINT THEOREMS ON COMPACT CONVEX SETS IN TOPOLOGICAL VECTOR SPACES, II

SEHIE PARK

In our previous work [7], we established fixed point theorems for multimaps $F: K \to 2^E$, where K is a nonempty compact convex subset of a topological vector space E having sufficiently many linear functionals (that is, E^* separates points of E). On the other hand, in a recent work [5], C.-W. Ha obtained multimap versions of fixed point theorems of Ky Fan [3, Theorems 1 and 3], which are comparable to the results in [7].

In the present paper, we give improvements and generalizations of some known results, mainly in [7] and [5]. In fact, Theorems 1 and 2 are improved versions of [7, Theorems 3 and 4]. Theorem 3 is a strengthened form of [5, Theorem 3] and equivalent to Reich [11, Theorem 2]. Theorem 4 generalizes [10, Theorem 3.1] and [5, Theorem 4]. Finally, in Theorem 5, we state sufficient conditions in order that a self-multimap $F: K \to 2^K$ have a fixed point. Consequently, as in [7], various generalizations of the Brouwer fixed point theorem are improved in this paper.

A t.v.s. stands for a Hausdorff topological vector space and an l.c.s. for a locally convex t.v.s. The notion of an upper semicontinuous (u.s.c.), a lower semicontinuous (l.s.c.), or an upper hemicontinuous (u.h.c.) multimap is standard, see [1], [7].

For a t.v.s. E and a $K \subset E$, the inward and outward sets of K at $x \in K$, $I_K(x)$ and $O_K(x)$, resp., are defined as follows:

$$I_K(x) := \{x + r(u - x) \in E : u \in K,\ r > 0\},$$
$$O_K(x) := \{x - r(u - x) \in E : u \in K,\ r > 0\}.$$

The closures of $I_K(x)$ and $O_K(x)$ are denoted by $\bar{I}_K(x)$ and $\bar{O}_K(x)$, resp. In the sequel, $W(x)$ denotes either $\bar{I}_K(x)$ or $\bar{O}_K(x)$.

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Let $cc(E)$ denote the family of nonempty closed convex subsets of E and $kc(E)$ the family of nonempty compact convex subsets.

For a t. v. s. E, let E^* denote its topological dual.

We begin with the following improved version of [7, Theorem 3].

THEOREM 1. *Let K be a nonempty compact convex subset of a t. v. s. E and F a continuous (i. e., u. s. c. and l. s. c.) multimap defined on K such that either*

(A) E^* separates points of E and $F : K \to kc(E)$, or

(B) E is locally convex and $F : K \to cc(E)$.

Then either F has a fixed point, or there exist a point $v \in K$ and a continuous seminorm p on E such that

$$0 < p(v - Fv) \leqq p(w - Fv) \text{ for all } w \in W(v),$$

where $p(w - Fv) = inf\{p(w - z) : z \in Fv\}$.

Proof. Use standard separation theorems [12], and just follow the proof of [7, Theorem 3].

Note that Reich [11, Theorem 3] follows from Theorem 1(B). Now we improve [7, Theorem 4] with the same proof.

THEOREM 2. *Under the hypothesis of Theorem 1, F has a fixed point if*

(1) for each $x \in Bd\,K \setminus Fx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that

$$|\lambda| < 1 \text{ and } (\lambda x + (1 - \lambda) Fx) \cap W(x) \neq \phi.$$

The following improved version of Ha [5, Theorem 3] can be compared with Theorem 1(B).

THEOREM 3. *Let K be a nonempty compact convex subset of a l. c. s. E, and $F : K \to kc(E)$ an u. s. c. multimap. Then either F has a fixed point, or there exist $v \in K$, $u_0 \in Fv$, and a continuous seminorm p on E such that*

$$0 < p(v - u_0) \leqq p(w - u_0) \text{ for all } w \in W(v).$$

Proof. Suppose that F has no fixed point. Then by [5, Theorem 3], there exist $v \in K$, $u_0 \in Fv$, and a continuous seminorm p on E such that

$$0 < p(v - u_0) \leqq p(w - u_0) \text{ for all } w \in K,$$

and hence, by the method in the proof of [7, Theorem 1], the inequality holds for all $w \in \bar{I}_K(v)$.

For the outward case, the map $F' : K \to kc(E)$ defined by $F'x = 2x - Fx$ for each $x \in K$ is u. s. c. Therefore, by the above argument, there exist $v \in K$, $u_1 \in F'v$, and a continuous seminorm p on E such that

$$0 < p(v - u_1) \leqq p(w' - u_1) \text{ for all } w' \in I_K(v).$$

For $w \in O_K(v)$, let $w' = 2v - w$ and $u_1 = 2v - u_0$ where $u_0 \in Fv$. Then we have

$$0 < p(v - u_0) \leqq p(w - u_0) \text{ for all } w \in O_K(v)$$

and hence, for all $w \in \bar{O}_K(v)$. This proves Theorem 3 for the case $W(v) = \bar{O}_K(v)$.

Note that Theorem 3 is equivalent to Reich [11, Theorem 2] which contains some known results in [3], [10], [2], and the Tychonoff fixed point theorem.

In certain circumstance, Theorem 3 is more useful than Theorem 1. Consider the following example due to Gwinner [4, p. 575]: Let $K = [0, 1] \times \{0\} \subset \boldsymbol{R}^2 \equiv E$ with the ordinary norm. Define $F : K \to 2^E$ by

$$F(c, 0) = \text{co}\{(1, 1), (1, 2)\} \text{ if } c \in [0, 1),$$
$$F(1, 0) = \text{co}\{(0, 0), (1, 1), (1, 2)\}.$$

Note that F is u. s. c., but not l. s. c., and Theorem 1 is not applicable. However, Theorem 3 holds for this example by choosing $v = (1, 0)$ and $u_0 = (1, 1)$.

As a direct consequence of Theorem 3, we have the following.

THEOREM 4. *Let K be a nonempty compact convex subset of a l. c. s. E, and $F : K \to kc(E)$ an u. s. c. multimap. Then F has a fixed point if*

(2) *for each $x \in Bd\, K \setminus Fx$ and $u \in Fx$, there exists a number λ (as in (1)) such that*

$$|\lambda| < 1 \textit{ and } \lambda x + (1 - \lambda) u \in W(x).$$

Proof. Suppose that F has no fixed point. By Theorem 3, there exist $v \in K$, $u_0 \in Fv$, and a continuous seminorm p on E such that

$$0 < p(v - u_0) \leqq p(w - u_0) \text{ for all } w \in W(v).$$

Since $v \notin Fv$, if $v \in \text{Int}\, K$, then there exists a λ (say $\lambda = 1/2$) such that $|\lambda| < 1$ and $\lambda v + (1 - \lambda) u_0 \in W(v) = E$. Therefore, there always exist a λ such that $|\lambda| < 1$ and $w := \lambda v + (1 - \lambda) u_0 \in W(v)$ whether $v \in \text{Int}\, K$

or $v \in \mathrm{Bd}\, K$. Then we have

$$0 < p(v-u_0) \leqq p(w-u_0) = |\lambda|\, p(v-u_0),$$

which contradicts $|\lambda| < 1$.

Note that (2) $\Rightarrow$ (1).

For a real t. v. s. E, (2) is equivalent to $Fx \subset W(x)$ for all $x \in K$, and hence there exist more general results than Theorem 4 (see [7, Theorems 6 and 7]). (Note here that the first part of the proof of [7, Theorem 6] showing the existence of $h \in E^*$ such that $(h, v) < 0$ for all $v \in u - Fu$ is incorrectly stated. Instead just use the separation theorem for a t. v. s. E on which E^* separates points (e. g., [12, p. 70]). Furthermore, in a recent work [8], we noted that [7, Theorem 6] holds for an u. h. c. map instead of an u. s. c. map.)

However, for a complex l. c. s. E, Theorem 4 generalizes Reich [10, Theorem 3. 1; 7, Theorem 5] and Ha [5, Theorem 4]. We note that these two results are the same. In fact, Reich adopted one of the following boundary conditions:

(3) for each $x \in \mathrm{Bd}\, K \setminus Fx$,

$$Fx \subset IF_K(x) = \{x + c(y-x) : y \in K,\ \mathrm{Re}(c) > 1/2\}.$$

(3)′ for each $x \in \mathrm{Bd}\, K \setminus Fx$,

$$Fx \subset OF_K(x) = \{x + c(y-x) : y \in K,\ \mathrm{Re}(c) < -1/2\}.$$

On the other hand, Ha used the following in Theorem 4 instead of (2):

(4) for each $x \in K$ and $u \in Fx$, there exists a number λ (as in (1)) such that

$$|\lambda| < 1 \text{ and } \lambda x + (1-\lambda) u \in K.$$

Note that (4) $\Rightarrow$ (2) since $K \subset I_K(x)$ and that $z \in IF_K(x)$ iff there is a number λ (as above) suh that $|\lambda| < 1$ and $\lambda x + (1-\lambda) z \in K$ [9]. Therefore, (3)$\Leftrightarrow$(4)$\Rightarrow$(2). Reich [9] noted that (3) can not be replaced by $Fx \cap IF_K(x) \neq \phi$. For a real t. v. s. E, (4) is equivalent to $Fx \subset I_X(x)$ for all $x \in K$.

Finally, for a selfmap $F : K \to 2^K$, we have the following improved version of [6, Theorem 1].

THEOREM 5. *Let K be a nonempty compact convex subset of a t. v. s. E having sufficiently many linear functionals, and $F : K \to cc(K)$. Then F has a fixed point if one of the following holds:*

(i) *F is continuous.*

(ii) *E is real and F is u. h. c.*

(iii) *E is locally convex and F is u. s. c.*

Proof. (i) follows from Theorem 2, (ii) is a consequence of the new version of [7, Theorem 6] in [8], and (iii) follows from Theorem 4.

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Seoul National University
and
The Mathematical Sciences Research Institute of Korea
Seoul 151-742, Korea