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NONLINEAR VARIATIONAL INEQUALITIES AND FIXED POINT THEOREMS

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1. Introduction

P. Hartman and G. Stampacchia [6] proved the following theorem in 1966: If $f: X \to R^n$ is a continuous map on a compact convex subset X of R^n, then there exists $x_0 \in X$ such that $\langle fx_0, x_0 - x\rangle \geq 0$ for all $x \in X$. This remarkable result has been investigated and generalized by F. E. Browder [1], [2], W. Takahashi [9], S. Park [8] and others. For example, Browder extended this theorem to a map f defined on a compact convex subset X of a topological vector space E into the dual space E^* ; see [2, Theorem 2]. And Takahashi extended Browder's theorem to closed convex sets in topological vector space; see [9, Theorem 3].

In Section 2, we obtain some variational inequalities, especially, generalizations of Browder's and Takahashi's theorems. The generalization of Browder's is an earlier result of the first author [8].

In Section 3, using Theorem 1, we improve and extend some known fixed point theorems. Theorems 4 and 8 improve Takahashi's results [9, Theorems 5 and 9], respectively. Theorem 4 extends the first author's fixed point theorem [8, Theorem 8] (Theorem 5 in this paper) which is a generalization of Browder [1, Theroem 1]. Theorem 8 extends Theorem 9 which is a generalization of Browder [2, Theorem 3].

Finally, in Section 4, we obtain variational inequalities for multi-valued maps by using Theorem 1. We improve Takahashi's results [9, Theorems 21 and 22] which are generalizations of Browder [2, Theorem 6] and the Kakutani fixed point theorem [7], respectively.

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2. Variational inequalities

Throughout this paper, we assume that a topological space is Hausdorff and a topological vector space is real. Let us start with the following useful theorem. We deduce this from the Brouwer fixed point theorem.

THEOREM 1. *Let X be a nonempty compact convex subset of a topological vector space E and f a real valued function on $X\times X$ satisfying*:

(i) *For each $y\in X$, the function $f(x, y)$ of x is lower semicontinuous*;
(ii) *for each $x\in X$, the function $f(x, y)$ of y is quasi-concave*; *and*
(iii) *$f(x, x)\leq c$ for all $x\in X$ with some real number c.*
Then there exists an $x_0\in X$ such that $f(x_0, y)\leq c$ for all $y\in X$.

Proof. Suppose that for each $x\in X$, there exists $y\in X$ such that $f(x, y)>c$. Then for each $y\in X$, the set $U_y=\{x\in X : f(x, y)>c\}$ is open by (i), and $\{U_y\}_{y\in X}$ is a cover of X. Since X is compact, there exists a finite family $\{y_1, y_2, \ldots, y_n\}$ such that $\{U_{y_i}\}_{i=1}^n$ covers X. Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a partition of unity subordinated to this subcover. Then each β_i is a continuous map of X into $[0, 1]$ which vanishes outside U_{y_i}, while $\sum_{i=1}^n \beta_i(x)=1$ for all $x\in X$. For each i satisfying $\beta_i(x)\neq 0$, x lies in U_{y_i}, so that $f(x, y_i)>c$. By (ii) we have

$$f(x, \sum_{i=1}^n \beta_i(x) y_i)>c$$

for all $x\in X$. Define a continuous map p of X into the convex hull of $\{y_1, y_2, \ldots, y_n\}$ by

$$p(x)=\sum_{i=1}^n \beta_i(x) y_i.$$

Since the convex hull of $\{y_1, y_2, \ldots, y_n\}$ is a compact convex subset of X which lies in a finite dimensional subspace of E, by the Brouwer fixed point theorem, we have $x_1\in X$ such that $x_1=p(x_1)=\sum_{i=1}^n \beta_i(x_1) y_i$. Hence we have

$$c\geq f(x_1, x_1)=f(x_1, \sum_{i=1}^n \beta_i(x_1) y_i)>c,$$

which is a contradiction.

Theorem 1 improves Takahashi [9, Lemma 1]. From Theorem 1, we obtain the following due to Fan [5] by setting $g(x, y)=f(x, x)-f(x, y)$ on $X\times X$.

COROLLARY 1. *Let X be a nonempty compact convex subset of a topological vector space E and f a real valued continuous function on $X\times X$ such that for each $x\in X$, the function $f(x, y)$ of y is quasi-convex. Then there exists $x_0\in X$ such that $f(x_0, x_0)\leq f(x_0, y)$ for all $y\in X$.*

Let X be a convex subset of a vector space E over R. For each $x\in X$, the inward and outward sets of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) := \{x+r(u-x)\in E : u\in X, r>0\},$$
$$O_X(x) := \{x-r(u-x)\in E : u\in X, r>0\}.$$

If E is a topological vector space, the closures of $I_X(x)$ and $O_X(x)$ are denoted by $\bar{I}_X(x)$ and $\bar{O}_X(x)$, respectively. In the sequel, $W(x)$ denotes either $\bar{I}_X(x)$ or $\bar{O}_X(x)$.

In [8], the first author obtained the following result by using Corollary 1.

COROLLARY 2. [8] *Let X be a nonempty compact convex subset of a topological vector space E and f a real valued continuous function on $X\times E$ such that for each $x\subset X$, the function $f(x, y)$ of y is convex. Then there exists an $x_0\in X$ such that $f(x_0, x_0)\leq f(x_0, y)$ for all $y\in\bar{I}_X(x_0)$.*

By using this, the first author proved the following:

THEOREM 2. [8] *Let X be a nonempty compact convex subset of a topological vector space E and f a continuous map of X into E^*. Then there exists an $x_0\in X$ such that $\langle fx_0, x_0-y\rangle\geq 0$ for all $y\in W(x_0)$.*

In particular, if E is locally convex and $W(x_0)$ is replaced by X, then Theorem 2 reduces to Browder [2, Theorem 2]. In [9, Theorem 3], Takahashi generalized Browder [2, Theorem 2] to closed convex sets in topological vector spaces. In the following theorem, we improve Takahashi's result. Let H, X be nonempty subsets of a topological vector space E. We put $B_HX=\bar{X}\cap\overline{H-X}$ and $I_HX=X\cap(B_HX)^c$ where $\bar{A}$ is the closure of $A\subset E$ and A^c is the complement of A.

THEOREM 3. *Let H be a closed convex subset of a topological vector space E and f a continuous map of H into E^*. If there exists a compact convex subset X of H such that $I_HX\neq\phi$ and for each $z\in B_HX$, there is $u_0\in I_HX$ with $\langle fz, z-u_0\rangle\geq 0$, then there exists $x^*\in H$ such that $\langle fx^*, y-x^*\rangle\geq 0$ for all $y\in\bar{I}_H(x^*)$.*

Proof. By Theorem 2, there exists $x^* \in X$ such that $\langle fx^*, y-x^* \rangle \geq 0$ for all $y \in I_X(x^*)$. If $x^* \in I_H X$, for each $y \in H$, we can choose λ $(0 < \lambda < 1)$ so that $x = x^* + \lambda(y-x^*)$ lies in X since the map $p(\lambda) = x^* + \lambda(y-x^*)$ is continuous. Then $y = x^* + (x-x^*)/\lambda$ lies in $I_X(x^*)$. Hence we obtain $\langle fx^*, y-x^* \rangle \geq 0$ for all $y \in H$. If $x^* \in B_H X$, by the hypothesis, there exists $u_0 \in I_H X$ with $\langle fx^*, x^*-u_0 \rangle \geq 0$. Since $\langle fx^*, x-x^* \rangle \geq 0$ for all $x \in X$, it follows that

$$\langle fx^*, x-u_0 \rangle \geq 0$$

for all $x \in X$. Since $u_0 \in I_H X$, for each $y \in H$ there exists λ $(0<\lambda<1)$ such that $x = u_0 + \lambda(y-u_0) \in X$. Hence

$$0 \leq \langle fx^*, x-u_0 \rangle = \lambda \langle fx^*, y-u_0 \rangle$$

and consequently $\langle fx^*, y-u_0 \rangle \geq 0$ for all $y \in H$. Since $u_0 \in X$ implies $\langle fx^*, u_0-x^* \rangle \geq 0$, we obtain $\langle fx^*, y-x^* \rangle \geq 0$ for all $y \in H$. For $y \in I_H(x^*) \setminus H$, $y = x^* + r(u-x^*)$ for some $u \in H$, $r>1$. So $\langle fx^*, y-x^* \rangle = r\langle fx^*, u-x^* \rangle \geq 0$. Hence $\langle fx^*, y-x^* \rangle \geq 0$ for all $y \in \bar{I}_H(x^*)$.

3. Fixed point theorems

In this section, using Theorem 1, we improve and extend some known fixed point theorems.

THEOREM 4. *Let X be a nonempty compact convex subset of a topological vector space E and f a continuous map of X into E. Then, either there exists $y_0 \in X$ such that y_0 and fy_0 cannot be separated by a continuous linear functional, or there exist $x_0 \in X$ and $g \in E^*$ such that*

$$g(x_0 - fx_0) < 0 \leq \inf_{y \in W(x_0)} g(x_0 - y).$$

Proof. Suppose that for each $x \in X$, there exists $h \in E^*$ such that $h(x-fx)<0$. Setting $U_h = \{x \in X : h(x-fx)<0\}$ for each $h \in E^*$, we have a cover $\{U_h\}_{h \in E^*}$ of X. Since X is compact, there exists a finite family $\{h_1, h_2, \ldots, h_n\}$ such that $\{U_{h_i}\}_{i=1}^n$ covers X. Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a partition of unity subordinated to this subcover. Define a real valued function p on $X \times E$ by

$$p(x, y) = \sum_{i=1}^n \beta_i(x) h_i(x-y).$$

Then, by Corollary 2, there exists $x_0 \in X$ such that

$$p(x_0, y) = \sum_{i=1}^n \beta_i(x_0) h_i(x_0-y) \geq p(x_0, x_0) = 0$$

for all $y \in \bar{I}_X(x_0)$. On the other hand, we have

$$p(x_0, fx_0) = \sum_{i=1}^n \beta_i(x_0) h_i(x_0-fx_0) < 0.$$

By putting $g=\sum_{i=1}^{n}\beta_i(x_0)h_i$, we obtain the desired result for inward case.

For outward case, define a continuous map $f':X\to E$ by $f'x=2x-fx$. Then, by the preceding inward case, either there exists $y_0\in X$ such that y_0 and $f'y_0$ cannot be separated by a continuous linear functional, or there exist $x_0\in X$ and $g'\in E^*$ such that $g'(x_0-f'x_0)<0\leq \inf_{z\in \bar{I}_X(x_0)} g'(x_0-z)$. The first alternative implies that y_0 and fy_0 cannot be separated by a continuous linear functional. Suppose that the second one holds. For any $y\in O_X(x_0)$, $z=2x_0-y$ lies in $I_X(x_0)$. Then we have

$$(-g')(x_0-fx_0)=(-g')(f'x_0-x_0)=g'(x_0-f'x_0)<0$$
$$\leq g'(x_0-z)=g'(y-x_0)=(-g')(x_0-y)$$

for any $y\in O_X(x_0)$, and hence for any $y\in \bar{O}_X(x_0)$. By putting $g=-g'$, we obtain the desired result for outward case.

Theorem 4 improves Takahashi [9, Theorem 5]. As a consequence of Theorem 4, we have the following:

THEOREM 5. [8] *Let X be a nonempty compact convex subset of a topological vector space E having sufficiently many linear functionals and f a continuous map of X into E. If for each $x\in X$, there exists $\lambda<1$ with $\lambda x+(1-\lambda)fx\in W(x)$, then f has a fixed point.*

Proof. Suppose f has no fixed point. By Theorem 4 there exist $x_0\in X$ and $g\in E^*$ such that

$$g(x_0-fx_0)<0\leq \inf_{y\in W(x_0)} g(x_0-y).$$

For this x_0, we can choose $\lambda<1$ with $y_0:=\lambda x_0+(1-\lambda)fx_0\in W(x_0)$. Hence we have

$$g(x_0-fx_0)<0\leq g(x_0-y_0)=(1-\lambda)g(x_0-fx_0).$$

This is a contradiction. Therefore f has a fixed point.

In particular, if E is locally convex and $W(x)$ is replaced by X, then Theorem 5 reduces to Browder [1, Theorem 1]. On the other hand, if f maps X into itself, we obtain the following:

COROLLARY 3. [3] *Let X be a nonempty compact convex subset of a topological vector space E having sufficiently many linear functionals and f a continuous map of X into itself. Then f has a fixed point.*

As another consequence of Theorem 4, we have the following:

THEOREM 6. *Let H be a closed convex subset of a topological vector space E having sufficiently many linear functionals and f a continuous map of H into H. If there exists a compact convex subset X of H such that for each $x \in B_H X$, there is $\lambda < 1$ with $\lambda x + (1-\lambda) fx \in \bar{I}_X(x)$, then f has a fixed point in H.*

Proof. Consider the restriction of f to X. If f has no fixed point in X, by Theorem 4 there exist $x_0 \in X$ and $g \in E^*$ such that

$$g(x_0 - fx_0) < 0 \leq \inf_{y \in W(x_0)} g(x_0 - y).$$

If $x_0 \in I_H X$, since $fx_0 \in H$, we can choose λ $(0 < \lambda < 1)$ so that $y_0 = \lambda x_0 + (1-\lambda) fx_0$ lies in X. Hence we have

$$g(x_0 - fx_0) < 0 \leq g(x_0 - y_0) = (1-\lambda) g(x_0 - fx_0).$$

This is a contradiction. If $x_0 \in B_H X$, by the hypothesis, there exists $\lambda < 1$ with $y_0 = \lambda x_0 + (1-\lambda) fx_0 \in \bar{I}_X(x_0)$. Also we have

$$g(x_0 - fx_0) < 0 \leq g(x_0 - y_0) = (1-\lambda) g(x_0 - fx_0),$$

which is a contradiction. Therefore f has a fixed point.

In particular, if E is locally convex and $W(x) = \bar{I}_X(x)$ is replaced by X, then Theorem 6 reduces to Takahashi [9, Theorem 7].

We now generalize Theorem 4 to multi-valued maps. The following definition is due to Fan [4]. Let X be a subset of a topological vector space E. A map T of X into 2^E is said to be upper demicontinuous if for each open half-space H in E, the set $\{x \in X : Tx \subset H\}$ is open in X. An open half-space H in E is a set of the form $\{x \in E : hx > r\}$ where h is a continuous linear functional, not identically zero, and r is a real number. It is obvious that if a map T of X into 2^E is upper semicontinuous, then T is upper demicontinuous. We say that two sets A, B in E can be strictly separated by a closed hyperplane, if there exist $h \in E^*$ and $r \in R$ such that $hx < r$ for all $x \in A$ and $hy > r$ for all $y \in B$. For two sets C, D in R, $C < D$ means that $x < y$ for any $x \in C$ and $y \in D$.

THEOREM 7. *Let X be a nonempty compact convex subset of a topological vector space E. Let S, T be two upper demicontinuous maps of X into 2^E such that for each $x \in X$, Sx and Tx are nonempty. Then, either there exists $y_0 \in X$ for which Sy_0 and Ty_0 cannot be strictly separated by a closed hyperplane, or there exist $x_0 \in X$ and $g \in E^*$ such that $g(x_0 - Tx_0) < g(x_0 - Sx_0)$ and $0 \leq \inf_{y \in W(x_0)} g(x_0 - y)$.*

Proof. Suppose that for each $x\in X$, Sx and Tx can be strictly separated by a closed hyperplane. Then for each $x\in X$, we can find $g_x\in E^*$ and $r_x\in R$ such that $g_x(Sx)<r_x<g_x(Tx)$. Since S, T are upper demicontinuous on X, there exists a neighborhood U_x of x in X such that $g_x(Sy)<r_x<g_x(Ty)$ for all $y\in U_x$. Hence x is in the interior $N(g_x)$ of $\{z\in X : g_x(Sz)<g_x(Tz)\}$. Thus $X=\bigcup_{x\in X} N(g_x)$. By compactness of X, there exists a finite set $\{x_1, x_2, \dots, x_n\}\subset X$ such that $X=\bigcup_{i=1}^{n} N(g_{x_i})$. Let $\{\beta_i\}_{i=1}^{n}$ be a partition of unity subordinated to the cover $\{N(g_{x_i})\}$. Define a real valued function p on $X\in E$ by

$$p(x, y)=\sum_{i=1}^{n}\beta_i(x)g_{x_i}(x-y).$$

By Corollary 2, there exists $x_0\in X$ such that

$$p(x_0, y)=\sum_{i=1}^{n}\beta_i(x_0)g_{x_i}(x_0-y)\geq p(x_0, x_0)=0$$

for all $y\in \bar{I}_X(x_0)$. We also know that

$$\sum_{i=1}^{n}\beta_i(x_0)g_{x_i}(Sx_0)<\sum_{i=1}^{n}\beta_i(x_0)g_{x_i}(Tx_0).$$

By putting $g=\sum\beta_i(x_0)g_{x_i}$, we obtain the desired result for inward case.

For outward case, define upper demicontinuous maps $S', T' : X\to 2^E$ by $S'x=2x-Sx$, $T'x=2x-Tx$, respectively. By the preceding inward case, either there exists $y_0\in X$ for which $S'y_0$ and $T'y_0$ cannot be strictly separated by a closed hyperplane, or there exist $x_0\in X$ and $g'\in E^*$ such that $g'(x_0-T'x_0)<g'(x_0-S'x_0)$ and $0\leq \inf_{z\in W(x_0)} g'(x_0-z)$. The first alternative implies that Sy_0 and Ty_0 cannot be strictly separated by a closed hyperplane. Suppose that the second one holds. For any $y\in O_X(x_0)$, $z=2x_0-y$ lies in $I_X(x_0)$. Then we have

$$\begin{aligned}(-g')(x_0-Tx_0)&=(-g')(T'x_0-x_0)=g'(x_0-T'x_0)\\ &<g'(x_0-S'x_0)=(-g')(S'x_0-x_0)=(-g')(x_0-Sx_0),\end{aligned}$$

and

$$0\leq g'(x_0-z)=g'(y-x_0)=(-g')(x_0-y)$$

for any $y\in O_X(x_0)$, and hence for any $y\in \bar{O}_X(x_0)$. By putting $g=-g'$, we obtain the desired result for outward case.

Theorem 7 improves Takahashi [9, Theorem 8]. If S is the identity map of X, then Theorem 7 reduces to the following generalization of Theorem 4.

THEOREM 8. *Let X be a nonempty compact convex subset of a topological vector space E and T an upper demicontinuous map of X into 2^E such that for each $x \in X$, Tx is nonempty. Then, either there exists $y_0 \in X$ such that y_0 and Ty_0 cannot be strictly separated by a closed hyperplane, or there exist $x_0 \in X$ and $g \in E^*$ such that*

$$g(x_0 - Tx_0) < 0 < \inf_{y \in W(x_0)} g(x_0 - y).$$

Theorem 8 improves Takahashi [9, Theorem 9]. As a consequence of Theorem 8, we have the following theorem.

THEOREM 9. *Let X be a nonempty compact convex subset of a locally convex topological vector space E and T an upper demicontinuous map X into 2^E such that for each $x \in X$, Tx is nonempty, closed and convex. If for each $x \in X$, there exists $\lambda < 1$ such that $(\lambda x + (1-\lambda)Tx) \cap W(x) \neq \phi$, then T has a fixed point.*

Proof. Suppose T has no fixed point. By Theorem 8 there exist $x_0 \in X$ and $g \in E^*$ such that

$$g(x_0 - Tx_0) < 0 \leq \inf_{y \in W(x_0)} g(x_0 - y).$$

For this x_0, we can choose $\lambda < 1$ and $z_0 \in Tx_0$ such that $y_0 := \lambda x_0 + (1-\lambda) z_0 \in W(x_0)$. Hence we have

$$g(x_0 - z_0) < 0 \leq g(x_0 - y_0) = (1-\lambda) g(x_0 - z_0).$$

This is a contradiction. Therefore T has a fixed point.

In particular, if T is upper semicontinuous and $W(x)$ is replaced by X, then Theorem 9 reduces to Browder [2, Theorem 3].

From Corollary 2 for a normed vector space, we obtain the following generalization of Ky Fan [4, Theorem 2].

THEOREM 10. *Let X be a nonempty compact convex subset of a normed vector space E and f a continuous map of X into E. Then there exists $x_0 \in X$ such that*

$$\|fx_0 - x_0\| = \min_{y \in W(x_0)} \|fx_0 - y\|.$$

Proof. Define a real valued function g on $X \times E$ by $g(x, y) = \|fx - y\|$. Then g is continuous and for each $x \in X$, the function $g(x, y)$ of y is convex. Thus the desired result is obvious by Corollary 2.

4. Variational inequalities for multi-valued maps

By using Theorem 1, we generalize Theorem 2 to multi-valued maps for inward case.

THEOREM 11. *Let X be a nonempty compact convex subset of a topological vector space E and T an upper semicontinuous map of X into 2^{E^*} such that for each $x\in X$, Tx is nonempty and compact. If for each $x\in X$,*

$$\min_{y\in X}\max_{g\in Tx}\langle g, x-y\rangle=\max_{g\in Tx}\min_{y\in X}\langle g, x-y\rangle,$$

then there exist $x_0\in X$ and $g_0\in Tx_0$ such that $\langle g_0, x_0-y\rangle\geq 0$ for all $y\in \bar{I}_X(x_0)$.

Proof. Define a real valued function f on $X\times X$ by

$$f(x, y)=\max_{g\in Tx}\langle g, x-y\rangle.$$

For any $y\in X$ and $c\in R$, put $A=\{x\in X : f(x, y)\geq c\}$. We show that if $\{x_\alpha : \alpha\in I\}$ is a net in A converging to x_0, then $x_0\in A$. For each x_α there exists $g_\alpha\in Tx_\alpha$ such that $\langle g_\alpha, x_\alpha-y\rangle\geq c$. Since $\cup\{Tx : x\in X\}$ is compact, $\{g_\alpha\}$ has a subnet $\{g_{\alpha'}\}$ converging to g_0. Since T is upper semicontinuous, $g_0\in Tx_0$. Also we have $c\leq \lim_{\alpha'}\langle g_{\alpha'}, x_{\alpha'}-y\rangle = \langle g_0, x_0-y\rangle$. Hence $x_0\in A$. That is, the function $f(x, y)$ of x is upper semicontinuous. It is obvious that the function $f(x, y)$ of y is convex and $f(x, x)=0$ for all $x\in X$. By Theorem 1 for $-f$, there exists $x_0\in X$ such that $\max_{g\in Tx_0}\langle g, x_0-y\rangle\geq 0$ for all $y\in X$. Since

$$\min_{y\in X}\max_{g\in Tx_0}\langle g, x_0-y\rangle=\max_{g\in Tx_0}\min_{y\in X}\langle g, x_0-y\rangle,$$

we have $g_0\in Tx_0$ such that $\langle g_0, x_0-y\rangle\geq 0$ for all $y\in X$. For $y\in I_X(x_0)\backslash X$, $y=x_0+r(u-x_0)$ for some $u\in X$ and $r>1$. So $\langle g_0, x_0-y\rangle= r\langle g_0, x_0-u\rangle\geq 0$. Hence $\langle g_0, x_0-y\rangle\geq 0$ for all $y\in \bar{I}_X(x_0)$.

Theorem 11 improves Takahashi [9, Theorem 21]. In particular, if Tx is convex, the minimax equality in Theorem 11 holds. So we have the following corollary which is an extension of Browder [2, Theorem 6].

COROLLARY 4. *Let X be a nonempty compact convex subset of a topological vector space $E$and T an upper semicontinuous map of X into 2^{E^*} such that for each $x\in X$, Tx is nonempty, compact and convex. Then there exist $x_0\in X$ and $g_0\in Tx_0$ such that $\langle g_0, x_0-y\rangle\geq 0$ for all $y\in \bar{I}_X(x_0)$.*

Proof. We need only show that for each $x\in X$,

$$\min_{y\in X}\max_{g\in Tx}\langle g, x-y\rangle=\max_{g\in Tx}\min_{y\in X}\langle g, x-y\rangle.$$

Let $x\in X$ and $c=\max_{g\in Tx}\min_{y\in X}\langle g, x-y\rangle$. For each $g\in Tx$, put $A(g)=\{y\in X : \langle g, x-y\rangle\leq c\}$. Let $\{g_1, g_2, \dots, g_n\}$ be a finite subset of Tx and $\{r_1, r_2, \dots, r_n\}$ be nonnegative numbers with $\sum_{i=1}^{n} r_i=1$. For $\sum_{i=1}^{n} r_i g_i\in Tx$, there is $y_0\in X$ such that $\sum_{i=1}^{n} r_i\langle g_i, x-y_0\rangle=\langle\sum_{i=1}^{n} r_i g_i, x-y_0\rangle\leq c$. Thus there exists $z\in X$ such that $\langle g_i, x-z\rangle\leq c$ for $i=1, 2, \dots, n$. Since the family $\{A(g) : g\in Tx\}$ has the finite intersection property and X is compact, we have $\cap\{A(g) : g\in Tx\}\neq\phi$. Let $y_0\in\cap\{A(g) : g\in Tx\}$. Then $\max_g\langle g, x-y_0\rangle\leq c$. Hence we have

$$\min_y \max_g\langle g, x-y\rangle\leq\max_g\langle g, x-y_0\rangle\leq\max_g \min_y\langle g, x-y\rangle.$$

On the other hand it is obvious that

$$\max_g \min_y\langle g, x-y\rangle\leq\min_y \max_g\langle g, x-y\rangle.$$

THEOREM 12. *Let X be a nonempty compact convex subset of Euclidean space R^n and T an upper semicontinuous map of X into 2^{R^n} such that for each $x\in X$, Tx is nonempty and compact. If for each $x\in X$,*

$$\min_{y\in X} \max_{z\in Tx}\langle z-x, x-y\rangle=\max_{z\in Tx} \min_{y\in X}\langle z-x, x-y\rangle,$$

then there exist $x_0\in X$ and $z_0\in Tx_0$ such that $\langle z_0-x_0, x_0-y\rangle\geq 0$ for all $y\in\bar{I}_X(x_0)$.

Proof. Setting $f(x, y)=\max_{z\in Tx}\langle z-x, x-y\rangle$ for $x, y\in X$ and applying the argument in Theorem 11, we obtain the desired result.

Theorem 12 improves Takahashi [9, Theorem 22].

COROLLARY 5. *Let X be a nonempty compact convex subset of Euclidean space R^n and T an upper semicontinuous map of X into 2^{R^n} such that for each $x\in X$, Tx is nonempty, compact and convex. Then there exist $x_0\in X$ and $z_0\in Tx_0$ such that $\langle z_0-x_0, x_0-y\rangle\geq 0$ for all $y\in\bar{I}_X(x_0)$.*

Proof. We need only show that for each $x\in X$,

$$\min_{y\in X} \max_{z\in Tx}\langle z-x, x-y\rangle=\max_{z\in Tx} \min_{y\in X}\langle z-x, x-y\rangle.$$

This follows from the argument in Corollary 4.

In Theorem 12, if T is a map of X into 2^X, by putting $y=z_0$, we obtain $z_0=x_0$, that is, $x_0\in Tx_0$. In particular, the Kakutani fixed point theorem is obtained.

COROLLARY 6. [7] *Let X be a nonempty compact convex subset of Eulidean space R^n and T an upper semicontinuous map of X into 2^X such that for each $x \in X$, Tx is nonempty, compact and convex. Then T has a fixed point.*

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