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Generalizations of Ky Fan's Matching Theorems and Their Applications

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We obtain generalizations of Ky Fan's matching theorems for open [or closed] coverings and their applications. Generalized forms of the KKM theorem, the Fan–Browder fixed point theorem, and the Schauder fixed point theorem and other new results are obtained.

1. INTRODUCTION

In [6], by using his own generalization [5] of the classical Knaster–Kuratowski–Mazurkiewicz theorem (simply, KKM theorem), Ky Fan obtained a matching theorem [6, Theorem 3] for open coverings of convex sets. From this, he also obtained another matching theorem [6, Theorem 2] for closed covertings of convex sets.

In the present paper, we first obtain a generalization of Ky Fan's matching theorem for closed coverings. Our result is equivalent to Lassonde [12, Theorem I], which is a generalization of the KKM theorem. We use Lassonde's theorem to obtain a far-reaching generalization of the Fan–Browder fixed point theorem [2] including results previously given by Takahashi [16], Lassonde [12], Ben-El-Mechaiekh, Deguire, and Granas [1], and Simons [15].

In the second part of this paper, we generalize Ky Fan's matching theorem for open coverings. Our new result is applied to recent results on open-valued KKM maps and fixed points due to W.K. Kim [9, 10] and some useful equivalent formulations.

Consequently, a number of generalizations of the KKM theorem, the Brouwer fixed point theorem, the Fan–Browder fixed point theorem and other new results are obtained.

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2. Preliminaries

We follow mainly Lassonde [12].

Let D and X be two sets. A *multifunction* $F: D \to 2^X$ is a function from D into the power set 2^X of X, that is, $Fx \subset X$ for each $x \in D$. Let $F(D) \equiv \bigcup \{Fx : x \in D\}$ and $F^{-1}y \equiv \{x \in D : y \in Fx\}$ for $y \in X$.

Let X be a convex set in a vector space and $D \subset X$. A multifunction $G: D \to 2^X$ is called *KKM* if $\mathrm{co}\{x_1, x_2, ..., x_n\} \subset \bigcup_{i=1}^n Gx_i$ for each finite subset $\{x_1, x_2, ..., x_n\} \subset D$, where co denotes the convex hull.

A *convex space* X is a nonempty convex set X (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. In fact, we may regard that X has the relative finite topology.

A nonempty subset L of a convex space X is called a *c-compact set* if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

A subset A of a topological space Y is said to be *compactly closed* [resp. *open*] in Y if for every compact set $K \subset Y$ the set $A \cap K$ is closed [resp. open] in K.

Let $\mathscr{C}(X, Y)$ denote the class of all continuous maps from X into Y.

The following version of Ky Fan's generalization [5] of the KKM theorem is due to Dugundji and Granas [3] and Lassonde [12]:

THEOREM 0. *Let D be a nonempty subset of a convex space X and $G: D \to 2^X$ a closed valued KKM multifunction. Then the family $\{Gx : x \in D\}$ has the finite intersection property. Further, if Gx is compact for some $x \in D$, then $\bigcap \{Gx : x \in D\} \neq \varnothing$.*

Note that if X is a convex subset of a Hausdorff topological vector space (*t.v.s.*), then the closedness assumption is the same to the relatively finitely closedness.

3. A Matching Theorem for Open Coverings

We begin with the following special case of the main result of this section.

THEOREM 1. *Let D be a nonempty subset in a compact convex space X, Y a topological space, and $A: D \to 2^Y$ a multifunction satisfying*

(i) *for each $x \in D$, Ax is compactly open in Y, and*

(ii) $A(D) = Y$.

Then, for each $s \in \mathscr{C}(X, Y)$, *there exist a nonempty finite subset* $\{x_1, x_2, ..., x_n\}$ *of* D *and an* $x_0 \in \operatorname{co}\{x_1, x_2, ..., x_n\}$ *such that* $sx_0 \in \bigcap_{i=1}^n Ax_i$.

Proof. For each $x \in D$, let $Fx \equiv Y \backslash Ax$ and let $Gx \equiv (s^{-1}F)x$. Suppose that the conclusion is false. Then for every finite subset $\{x_1, x_2, ..., x_n\}$ of D, we have

$$s(\operatorname{co}\{x_1, x_2, ..., x_n\}) \subset Y \diagdown \bigcap_{i=1}^{n} Ax_i = \bigcup_{i=1}^{n} Fx_i,$$

that is,

$$\operatorname{co}\{x_1, x_2, ..., x_n\} \subset \bigcup_{i=1}^{n} (s^{-1}F)x_i = \bigcup_{i=1}^{n} Gx_i.$$

Therefore, G is a closed valued KKM map. Since X is compact, each Gx is compact. Then by Theorem 0, we have $\bigcap\{Gx : x \in D\} = \bigcap\{(s^{-1}F)x : x \in D\} \neq \emptyset$, and hence $\bigcap\{Fx : x \in D\} \neq \emptyset$. But this means $\bigcup\{Ax : x \in D\} = A(D) \neq Y$ against our hypothesis. This completes our proof.

Remark. If X is contained in a convex subset Y of a topological vector space E and if s is the inclusion of X into Y, then Theorem 1 reduces to [6, Lemma 1].

The following consequence of Theorem 1 is the main result of this section.

THEOREM 2. *Let* D *be a nonempty subset of a convex space* X, Y *a topological space, and* $A : D \to 2^Y$ *a multifunction satisfying*

(i) *for each* $x \in D$, Ax *is compactly open in* Y,

(ii) $A(D) = Y$, *and*

(iii) *there are a c-compact set* $L \subset X$ *and a compact set* $K \subset Y$ *such that* $Y \backslash A(L \cap D) \subset K$.

Then, for each $s \in \mathscr{C}(X, Y)$, *there exist a nonempty finite subset* $\{x_1, x_2, ..., x_n\}$ *of* D *and an* $x_0 \in \operatorname{co}\{x_1, x_2, ..., x_n\}$ *such that* $sx_0 \in \bigcap_{i=1}^n Ax_i$.

Proof. In the case when $A(L \cap D) = Y$, the conclusion follows from Theorem 1. Let $Y \backslash A(L \cap D)$ be nonempty. Since $A(D) = Y \supset K \supset Y \backslash A(L \cap D)$, we obtain $\{x_1, x_2, ..., x_n\} \subset D \backslash L$ such that $Y \backslash A(L \cap D) \subset \bigcup_{i=1}^n Ax_i$. Consider $D_1 \equiv (L \cap D) \cup \{x_1, x_2, ..., x_n\}$ and $X_1 \equiv \operatorname{co}(L \cup \{x_1, x_2, ..., x_n\})$. Since $D_1 \subset D$, $D_1 \subset X_1 \subset X$, $A(D_1) = Y$, and X_1 is compact, we obtain the conclusion by again using Theorem 1.

Remarks. 1. We modified the proof of Takahashi [16, Theorem 1].

2. If $X \equiv Y$ is a convex subset of a t.v.s. E and $s \equiv 1_X$, Theorem 2 reduces to Ky Fan [6, Theorem 3].

3. In fact, (iii) implies that $Y \backslash A(L \cap D)$ is compact. Observe that in case X or Y is compact the condition (iii) is immediately fulfilled; more precisely, the case of a compact X reduces to the case of a compact Y by considering $Y' \equiv sX$ and $A'x = Ax \cap Y'$.

4. Generalizations of the KKM Theorem

Theorem 2 may be restated in its contrapositive form and in terms of the complement Fx of Ax in Y as follows:

THEOREM 3. *Let D be a nonempty subset of a convex space X, Y a topological space, $F: D \to 2^Y$ a multifunction, and $s \in \mathscr{C}(X, Y)$ satisfying*

(i) *for each $x \in D$, Fx is compactly closed in Y,*

(ii) *for any finite subset $\{x_1, x_2, ..., x_n\}$ of D,*

$$s(\mathrm{co}\{x_1, x_2, ..., x_n\}) \subset \bigcup_{i=1}^{n} Fx_i,$$

and

(iii) *there are a c-compact set $L \subset X$ and a compact set $K \subset Y$ such that $\bigcap \{Fx : x \in L \cap D\} \subset K$.*

Then we have $\bigcap \{Fx : x \in D\} \neq \varnothing$.

Remarks. 1. Theorem 3 is due to Lassonde [12, Theorem I].

2. If $X \equiv Y$ and $s \equiv 1_X$, Theorem 3 reduces to Ky Fan [6, Theorem 4].

3. If L is a singleton, then we have the following generalization of Theorem 0.

THEOREM 4. *Let D be a nonempty subset of a convex space X, Y a topological space, $F: D \to 2^Y$ a multifunction, and $s \in \mathscr{C}(X, Y)$ satisfying* (i) *and* (ii) *of Theorem* 3, *and*

(iii) *Fx is compact for at least one $x \in D$.*

Then we have $\bigcap \{Fx : x \in D\} \neq \varnothing$.

Remark. In fact, Theorems 0–4 are all logically equivalent to the KKM theorem, the Sperner lemma, and the Brouwer fixed point theorem.

For certain applications, it is convenient to reformulate Theorem 3 in the following form.

THEOREM 5. *Let D be a nonempty subset of a convex space X, Y a topological space, $B \subset X \times Y$, and $s \in \mathscr{C}(X, Y)$ satisfying*

(i) *for each $x \in D$, $\{y \in Y: (x, y) \in B\}$ is compactly closed in Y, and*

(ii) *for any finite subset $\{x_1, x_2, ..., x_n\}$ of D and for any n positive reals α_i with $\sum_{i=1}^n \alpha_i = 1$, we have $(x_k, s(\sum_{i=1}^n \alpha_i x_i)) \in B$ for some k, $1 \leqslant k \leqslant n$.*

Then for any c-compact set $L \subset X$, either there exists $y_1 \in Y$ such that $(x, y_1) \in B$ for all $x \in D$, or for any compact set $K \subset Y$, there exists $y_2 \in Y \backslash K$ such that $(x, y_2) \in B$ for all $x \in L \cap D$.

Remark. If $X \equiv Y$, $K \equiv L$, and $s \equiv 1_X$, Theorem 5 reduces to an earlier result of Ky Fan.

5. GENERALIZATIONS OF THE FAN–BROWDER FIXED POINT THEOREM

As an application of Theorem 2, we give the following generalization of the Fan–Browder fixed point theorem [2].

THEOREM 6. *Let D be a nonempty subset of a convex space X and Y a topological space. Let $A, B: D \to 2^Y$ be multifunctions satisfying the following conditions:*

(a) *$Bx \subset Ax$ for each $x \in D$,*

(b) *$A^{-1}y$ is convex for each $y \in Y$,*

(c) *$B^{-1}y \neq \varnothing$ for each $y \in Y$,*

(d) *Bx is compactly open for each $x \in D$, and*

(e) *for some c-compact set $L \subset X$, $Y \backslash B(L \cap D)$ is compact.*

Then, for any $s \in \mathscr{C}(X, Y)$, there exists an $x_0 \in D$ such that $sx_0 \in Ax_0$.

Proof. Since the conditions (d), (c), and (e) imply (i), (ii), and (iii) of Theorem 2, respectively, there exist $\{x_1, x_2, ..., x_n\} \subset D$ and $x_0 \in \text{co}\{x_1, x_2, ..., x_n\}$ such that $sx_0 \in \bigcap_{i=1}^n Bx_i \subset \bigcap_{i=1}^n Ax_i$. We have $x_i \in A^{-1}(sx_0)$ for all $i = 1, ..., n$, and hence, by (b), $\text{co}\{x_1, x_2, ..., x_n\} \subset A^{-1}(sx_0)$. In particular, $x_0 \in A^{-1}(sx_0)$, that is, $sx_0 \in Ax_0$. This completes our proof.

Remarks. 1. The condition (c) can be replaced by the following:

(c)′ $B^{-1}y \neq \varnothing$ for each $y \in Y \backslash B(L \cap D)$.

For, given $y \in B(L \cap D)$, there exists an $x \in L \cap D$ such that $y \in Bx$, that is, $x \in B^{-1}y$.

2. Theorem 6 is the main result of our previous work [13], where various applications of Theorem 6 are given.

3. As we noted in [13], Theorem 6 generalizes Takahashi [16, Theorems 2 and 5], Lassonde [12, Theorem 1.1], Ben-El-Mechaiekh, Deguire, and Granas [1, Theorem 1], Simons [15, Theorem 4.3], Ko and Tan [11, Theorem 3.1], and Browder [2, Theorem 1].

6. A Matching Theorem for Closed Coverings

We begin with the following main theorem in this section.

THEOREM 7. *Let X be a convex space, Y a topological space, and $s \in \mathscr{C}(X, Y)$. Let $A_1, A_2, ..., A_n$ be a finite family of n compactly closed subsets of Y such that $\bigcup_{i=1}^n A_i = Y$. Then for any n points $x_1, x_2, ..., x_n$ (not necessarily distinct) of X, there exist k indices (for a suitable k) $i_1 < i_2 < \cdots < i_k$ such that*

$$s(\mathrm{co}\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}) \cap \bigcap_{j=1}^{k} A_{i_j} \neq \varnothing.$$

Proof. Consider an $(n-1)$-simplex $\Delta^{n-1} = b_1 b_2 \cdots b_n$ and define $f: \Delta^{n-1} \to X$ by $f(\sum_{i=1}^n \lambda_i h_i) = \sum_{i=1}^n \lambda_i x_i$ for any $\lambda_i \geqslant 0$ with $\sum_{i=1}^n \lambda_i = 1$. For each $z \in \Delta^{n-1}$, let $I(z) = \{i: (sf)z \in A_i\} \neq \varnothing$, and let Tz be the convex hull of $\{b_i: i \in I(z)\}$. For each $z \in \Delta^{n-1}$, $\bigcup \{A_i: i \notin I(z)\}$ is compactly closed so $U_z = \Delta^{n-1} \setminus f^{-1}s^{-1}(\bigcup \{A_i: i \notin I(z)\})$ is an open neighborhood of z in Δ^{n-1}. If $z' \in U_z$, then $I(z') \subset I(z)$ and therefore $Tz' \subset Tz$. Consequently, T is an u.s.c. multifunction defined on Δ^{n-1}. For each $z \in \Delta^{n-1}$, Tz is a nonempty compact convex subset of Δ^{n-1}. Therefore, by the Kakutani fixed point theorem, there exists a point $z \in \Delta^{n-1}$ such that $z \in Tz$. If we take $J = \{x_i: i \in I(z)\}$, then $x_0 \equiv fz \in \mathrm{co}\, J$ and $sx_0 \in \bigcap \{A_i: i \in I(z)\}$. This completes our proof.

For $X \equiv Y$ and $s \equiv 1_X$, Theorem 7 reduces to the following matching theorem of Ky Fan [6, Theorem 2] for closed coverings of convex sets.

COROLLARY 7.1. *Let X be a convex space and $\{A_i\}_{i=1}^n$ a finite family of n closed subsets of X such that $\bigcup_{i=1}^n A_i = X$. Then for any n points $x_1, x_2, ..., x_n$ of X, there exist k indices $i_1 < i_2 < \cdots < i_k$ such that*

$$\mathrm{co}\{x_{i_1}, x_{i_2}, ..., x_{i_k}\} \cap \bigcap_{j=1}^{k} A_{i_j} \neq \varnothing.$$

Remark. Ky Fan obtained Corollary 7.1 from [6, Theorem 1], which is a consequence of the well-known Fan–Glicksberg fixed point theorem [4, 7], and also from Theorem 2. However, we obtained Theorem 7, and hence Corollary 7.1, from the Kakutani fixed point theorem [8].

From Theorem 7 we obtain generalizations of a KKM type theorem which is an open valued version of the first part of Theorem 0.

THEOREM 8. *Let D be a nonempty subset of a convex space X, Y a topological space, $G: D \to 2^Y$ a multifunction, and $s \in \mathscr{C}(X, Y)$. Suppose that*

(i) *for each $x \in D$, Gx is compactly open in Y, and*

(ii) *for any finite subset $\{x_1, x_2, ..., x_n\}$ of D,*

$$s(\mathrm{co}\{x_1, x_2, ..., x_n\}) \subset \bigcup_{i=1}^{n} Gx_i.$$

Then the family $\{Gx: x \in D\}$ has the finite intersection property.

Proof. Suppose, on the contrary, that there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of D such that $\bigcap_{i=1}^n Gx_i = \varnothing$. Let $F: D \to 2^Y$ be given by $Fx = X \backslash Gx$ for each $x \in D$. Then $\{Fx_i\}_{i=1}^n$ is a family of compactly closed subsets of Y such that

$$\bigcup_{i=1}^{n} Fx_i = Y \Big\backslash \bigcap_{i=1}^{n} Gx_i = Y.$$

Therefore, by Theorem 7, there exist k indices $i_1 < i_2 < \cdots < i_k$ such that

$$s(\mathrm{co}\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}) \cap \bigcap_{j=1}^{k} Fx_{i_j} \neq \varnothing,$$

that is,

$$s(\mathrm{co}\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}) \not\subset Y \Big\backslash \bigcap_{j=1}^{k} Fx_{i_j} = \bigcup_{j=1}^{k} Gx_{i_j},$$

which contradicts the condition (ii). This completes our proof.

For $X \equiv Y$ and $s \equiv 1_X$, Theorem 8 reduces to the following:

COROLLARY 8.1. *Let D be a nonempty subset of a convex space X and $G: D \to 2^X$ a compactly open valued KKM multifunction. Then the family $\{Gx: x \in D\}$ has the finite intersection property.*

Remarks. 1. If X is a convex subset of a t.v.s. E, then the openness of Gx is actually relatively finitely open. Therefore, in this case, Corollary 8.1 generalizes W. K. Kim [10, Theorems 1 and 2].

2. If X is a t.v.s. E, Corollary 8.1 generalizes W. K. Kim [9, Theorem 2].

3. The following consequence of Corollary 8.1 improves W. K. Kim [9, Theorem 1].

COROLLARY 8.2. *Let $\Delta^{n-1} = b_1 b_2 \cdots b_n$ be an $(n-1)$-simplex in $E = \mathbb{R}^n$ and $G: \{b_1, b_2, ..., b_n\} \to E$ be a compactly open valued KKM map. Then $\bigcap_{i=1}^{n} Gx_i \neq \varnothing$.*

7. FIXED POINT THEOREMS

In this final section, we obtain new types of fixed point theorems from Theorem 7 and some of their equivalent formulations.

The following is a coincidence result comparable to the generalized Fan–Browder fixed point theorem (Theorem 6).

THEOREM 9. *Let X be a convex space, Y a topological space, and $A, B: X \to 2^Y$ multifunctions satisfying*

(a) *$Bx \subset Ax$ for each $x \in X$,*

(b) *$A^{-1}y$ is convex for each $y \in Y$,*

(c) *Bx is compactly closed for each $x \in X$, and*

(d) *there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $Y = \bigcup_{i=1}^{n} Bx_i$.*

Then, for any $s \in \mathscr{C}(X, Y)$, there exists an $x_0 \in X$ such that $sx_0 \in Ax_0$.

Proof. By Theorem 7, there exist a subset $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$ and an $x_0 \in \mathrm{co}\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}$ such that $sx_0 \in \bigcap_{j=1}^{k} Bx_{i_j} \subset \bigcap_{j=1}^{k} Ax_{i_j}$. Hence, $sx_0 \in Ax_{i_j}$ for all j, that is, $x_{i_j} \in A^{-1}(sx_0)$. Since $A^{-1}(sx_0)$ is convex and x_0 is a convex combination of x_{i_j}'s, we have $x_0 \in A^{-1}(sx_0)$, that is, $sx_0 \in Ax_0$.

Remark. For $X \equiv Y$, $A \equiv B$, and $s \equiv 1_X$, Theorem 9 reduces to W. K. Kim [10, Theorem 4; 9, Theorem 3].

The following is a dual form of Theorem 9.

THEOREM 10. *Let X be a topological space, Y a convex space, and $S, T: X \to 2^Y$ multifunctions satisfying*

(a) *$Tx \subset Sx$ for each $x \in X$,*

(b) *Sx is convex for each $x \in X$,*

(c) *$T^{-1}y$ is compactly closed for each $y \in Y$, and*

(d) *there exists a finite subset $\{y_1, y_2, ..., y_n\}$ of Y such that $X = \bigcup_{i=1}^{n} T^{-1}y_i$.*

Then, for any $f \in \mathscr{C}(Y, X)$, there exists a $y_0 \in Y$ such that $y_0 \in (Sf)\, y_0$.

Proof. Define $A, B\colon Y \to 2^X$ by $Ay \equiv S^{-1}y$ and $By = T^{-1}y$ for $y \in Y$. Noting that $A^{-1}x = Sx$ and $B^{-1}x = Tx$ for each $x \in X$, it is easy to see that A and B satisfy the conditions of Theorem 9 interchanging X and Y. Therefore, by Theorem 9, for any $f \in \mathscr{C}(Y, X)$, there exists a $y_0 \in Y$ such that $fy_0 \in Ay_0 = S^{-1}y_0$, that is, $y_0 \in (Sf)\, y_0$. This completes our proof.

For $X \equiv Y$, $S \equiv T$, and $s \equiv 1_X$, we have the following:

COROLLARY 10.1. *Let X be a convex space and $T\colon X \to 2^X$ an u.s.c. multifunction such that*

(1) *Tx is convex for each $x \in X$, and*

(2) *X can be covered by a finite number of $T^{-1}x$'s.*

Then T has a fixed point.

Proof. Note that each $T^{-1}x$ is closed since T is u.s.c.

Remark. W. K. Kim [10, Corollary 1; 9, Theorem 4] assumed that, instead of (2), there exists a finite subset K of X such that $Tx \cap K \neq \varnothing$ for every $x \in X$, and obtained Corollary 10.1.

In fact, let $K = \{x_1, x_2, ..., x_n\}$. Then for any $x \in X$, there is an $x_i \in Tx$, whence $x \in T^{-1}x_i$.

The following geometric forms of Theorem 9 are easily seen to be equivalent to each other.

THEOREM 11. *Let X be a convex space, Y a topological space, and $A, B \subset X \times Y$ sets satisfying*

(i) *$B \subset A$,*

(ii) *for each $y \in Y$, $\{x \in X : (x, y) \in A\}$ is convex,*

(iii) *for each $x \in X$, $\{y \in Y : (x, y) \in B\}$ is compactly closed, and*

(iv) *there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that for any $y \in Y$, $(x_i, y) \in B$ for some i, $1 \leqslant i \leqslant n$.*

Then, for any $s \in \mathscr{C}(X, Y)$, there exists an $x_0 \in X$ such that $(x_0, sx_0) \in A$.

THEOREM 12. *Let X be a convex space, Y a topological space, and $B \subset A \subset Z$ sets. Let $g: X \times Y \to Z$ be a function satisfying*

(1) *for each $y \in Y$, $\{x \in X: g(x, y) \in A\}$ is convex,*

(2) *for each $x \in X$, $\{y \in Y: g(x, y) \in B\}$ is compactly closed, and*

(3) *there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that for any $y \in Y$, $g(x_i, y) \in B$ for some i, $1 \leqslant i \leqslant n$.*

Then, for any $s \in \mathscr{C}(X, Y)$, there exists an $x_0 \in X$ such that $g(x_0, sx_0) \in A$.

The following alternative for two functions is useful.

THEOREM 13. *Let X be a convex space, Y a topological space, and $\alpha \geqslant \beta$. Let $f, g: X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions satisfying*

(i) *$g(x, y) \leqslant f(x, y)$ for each $(x, y) \in X \times Y$,*

(ii) *for each $y \in Y$, $\{x \in X: f(x, y) \geqslant \beta\}$ is convex, and*

(iii) *for each $x \in X$, $\{y \in Y: g(x, y) \geqslant \alpha\}$ is compactly closed.*

Then either

(1) *for any finite subsets $\{x_1, x_2, ..., x_n\}$ of X, there exists a $y_0 \in Y$ such that $g(x_i, y_0) < \alpha$ for all i, $1 \leqslant i \leqslant n$, or*

(2) *for any $s \in \mathscr{C}(X, Y)$, there exists an $x_0 \in X$ such that $f(x_0, sx_0) \geqslant \beta$.*

Proof. For each $x \in X$, let

$$Ax \equiv \{y \in Y: f(x, y) \geqslant \beta\},$$
$$Bx \equiv \{y \in Y: g(x, y) \geqslant \alpha\}.$$

Then by Theorem 9, we have the conclusion.

Moreover, we can obtain Theorem 9 from Theorem 13.

Proof of Theorem 9 using Theorem 13. Just choose $\alpha = \beta = 1$ and $f, g: X \times Y \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} 1 & \text{if } y \in Bx, \\ 0 & \text{if } y \notin Bx, \end{cases} \qquad f(x, y) = \begin{cases} 1 & \text{if } y \in Ax, \\ 0 & \text{if } y \notin Ax. \end{cases}$$

Remark. In Theorem 13, the conditions (ii) and (iii), resp., can be replaced by the following:

(ii)′ $f(\cdot, y)$ is quasi-concave (that is, $\{x \in X: f(x, y) > \alpha\}$ is convex for any $\alpha \in \mathbb{R}$).

(iii)′ $g(x, \cdot)$ is u.s.c. on compact subsets of Y (that is, $\{y \in Y: g(x, y) \geqslant \alpha\}$ is compactly closed for any $\alpha \in \mathbb{R}$).

The dual form of Theorem 13 in the case where $X \equiv Y$ and $f \equiv g$ can be stated as follows:

COROLLARY 13.1. *Let X be a convex space, $\alpha \geqslant \beta$, and $g: X \times X \to \mathbb{R} \cup \{+\infty\}$ a function satisfying*

(a) *for each $y \in X$, $\{x \in X: g(x, y) \leqslant \beta\}$ is convex,*

(b) *for each $x \in X$, $\{y \in X: g(x, y) \leqslant \alpha\}$ is compactly closed, and*

(c) *there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that for any $y \in X$, $g(x_i, y) \leqslant \alpha$ for some i, $1 \leqslant i \leqslant n$.*

Then, for any $s \in \mathscr{C}(X, X)$, there exists an $x_0 \in X$ such that $g(x_0, sx_0) \leqslant \beta$.

Remark. In Corollary 13.1, the conditions (a) and (b) resp., can be replaced by the following:

(a)′ $g(\cdot, y)$ is quasi-convex.

(b)′ $g(x, \cdot)$ is l.s.c. on compact subsets of X.

From Corollary 13.1, we obtain a fixed point theorem:

COROLLARY 13.2. *Let (X, d) be a metric compact convex space such that every ball is convex. Then every $f \in \mathscr{C}(X, X)$ has a fixed point.*

Proof. Since $\{y \in X: d(x, y) \leqslant \varepsilon\}$ is convex and closed for any $x \in X$ and $\varepsilon > 0$. Since X is compact, it is covered by a finite number of open ε-balls, and hence by closed ε-balls. Therefore, by Corollary 13.1, we have an $x_\varepsilon \in X$ such that $d(x_\varepsilon, fx_\varepsilon) \leqslant \varepsilon$. Since X is compact, we may assume that for some sequence $\varepsilon_n \to 0$ we have $x_{\varepsilon_n} \to y \in X$. Since f is continuous, we have $fy = y$.

Remarks. 1. In a normed space, every ball is convex. Therefore, Corollary 13.2 generalizes the Schauder fixed point theorem [14].

2. We have deduced Theorems 7–13 and Corollary 13.2 from the Kakutani fixed point theorem, which is logically equivalent to the Brouwer fixed point theorem. Therefore, all of Theorems 1–13 and Corollaries 13.1 and 13.2 are logically equivalent to the KKM theorem, the Sperner lemma, and the Brouwer fixed point theorem.

Finally, from Theorem 13, we have the following:

COROLLARY 13.3. *Let X be a convex space and $f: X \times X \to \mathbb{R}$ a function satisfying*

(i) *for each $y \in X$, $f(\cdot, y)$ is quasi-concave,*

(ii) *for each $x \in X$, $f(x, \cdot)$ is u.s.c. on compact subsets of X, and*

(iii) *for any* $\alpha \in \mathbb{R}$, *there exists a finite subset* $\{x_1, x_2, ..., x_n\}$ *of* X *such that for any* $y \in X$, $f(x_i, y) \geqslant \alpha$ *for some* i, $1 \leqslant i \leqslant n$.

Then the restriction of f *to the diagonal of* $X \times X$ *is not bounded from above.*

Proof. For any $\alpha \in \mathbb{R}$, by Theorem 13, there exists an $x \in X$ such that $f(x, x) \geqslant \alpha$.

COROLLARY 13.4. *Let* X *be a compact convex space and* $f: X \times X \to \mathbb{R}$ *a function satisfying*

(i) *for each* $y \in X$, $f(\cdot, y)$ *is quasi-concave,*

(ii) *for each* $x \in X$, $f(x, \cdot)$ *is continuous, and*

(iii) *for any* $\alpha \in \mathbb{R}$, *there exist* $x, y \in X$ *such that* $f(x, y) > \alpha$.

Then the restriction of f *to the diagonal of* $X \times X$ *is not bounded from above.*

Proof. Let $Ax \equiv \{y \in X: f(x, y) > \alpha\}$. Then $\{Ax: x \in X\}$ is an open covering of X. Therefore, the condition (iii) of Corollary 13.3 holds.

Remark. Corollary 13.4 includes W. K. Kim [10, Theorem 5; 9, Corollary 2].

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