Bull. Korean Math. Soc. 24(1987), No. 2, pp. 127-129

SOME APPROXIMATIONS OF SOLUTIONS OF THE LIONS-STAMPACCHIA VARIATIONAL INEQUALITY

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1. Introduction.

Let K be a closed convex subset of a Hilbert space H. If $A \in L(H, H^*)$ is a bounded linear maps from H into H^*, we consider the following variational inequality: For any given $f \in H^*$,

(*) (1) $x \in K$

(2) $(Ax, x-y) \leq (f, x-y)$ for any $y \in K$.

If $A=J$ is the duality map and P is the nearest projection map from H to K, then the above variational inequality has a solution $x=PJ^{-1}f$. Indeed, the nearest projection map P is characterized by the following property:

$$z \in K \text{ and } (y-z, x-z) \leq 0 \text{ for all } y \in K \text{ iff } z=Px.$$

More generally, Lions and Stampacchia [1] proved the existence of the unique solution of the above variational inequality for H-elliptic A.

In this paper, we apply the following theorem which is a geometric estimation about fixed points for contractions in a Hilbert space to the whereabout of the solution of the above variational inequality.

THEOREM [2, 3]. *Let H be a Hilbert space and $F: D \subset H \to H$ a contraction with a Lipschitzian constant $L<1$. If* Fix $F \neq \phi$, $x_0 \in D$, *then either*

(1) $Fx_0 = x_0$

or (2) Fix $F \subset \overline{B}(m, r) \cap D \setminus B(x_0, s)$

where $s = \|x_0 - Fx_0\|(1+L)^{-1}$,

$m = (1-(1-L^2)^{-1})x_0 + (1-L^2)^{-1}Fx_0$

and $r = \|x_0 - Fx_0\| L (1-L^2)^{-2}$.

Received March 23, 1987.
Supported by a grant from the Korea Science and Engineering Foundation, 1986-87.

2. Main Result.

We apply the above theorem to the result of Lions and Stampacchia and obtain the whereabouts of the unique solution of the variational inequality (*). For any given $f \in H^*$ in (*) we define a map $F: K \to K$ as $Fx = P(x - \rho J^{-1}(Ax - f))$ for all $x \in K$, where J is the duality map and $\rho > 0$. By using the map F we find a localization of the solution of (*) in the following:

THEOREM. *Suppose $A \in L(H, H^*)$ is H-elliptic: there exists $c > 0$ such that for any $x \in H$, $(Ax, x) \geq c\|x\|^2$. If we choose $\rho = c\|A\|^{-2}$, then F has the unique fixed point in K which is the unique solution in (*). In case that Fx_0 is not a solution of (*), $x_0 \in K$, the solution of (*) is in $\overline{B}(m, r) \setminus B(x_0, s)$ where $s = \|x_0 - Fx_0\|(1+L)^{-1}$, $L^2 = 1 - c\rho$, $m = (1 - (c\rho)^{-1})x_0 + (c\rho)^{-1}Fx_0$ and $r = \|x_0 - Fx_0\|(1 - c\rho)^{\frac{1}{2}}(c\rho)^{-2}$.*

Proof. The variational inequality (*) is restated as the following conditions: For any y in K,

$$
\begin{aligned}
& (Ax - f,\ x - y) \leq 0 \\
\text{iff} \quad & (J^{-1}(Ax - f),\ x - y) \leq 0 \\
\text{iff} \quad & (\rho J^{-1}(Ax - f),\ x - y) \leq 0 \quad \text{for any} \quad \rho > 0 \\
\text{iff} \quad & (x - x + \rho J^{-1}(Ax - f),\ x - y) \leq 0 \quad \text{for any} \quad \rho > 0.
\end{aligned}
$$

By the characterization of the nearest projection P on K we have the following equivalent condition for the solution x in (*):

$$x = P(x - \rho J^{-1}(Ax - f)).$$

That is, x is the fixed point of F. Therefore, if we may choose $\rho > 0$ so small that F is a contraction with a Lipschitzian constant $L < 1$, then we apply the above theorem to this setting. We estimate

$$
\begin{aligned}
\|Fx - Fy\| &= \|P(x - \rho J^{-1}(Ax - f)) - P(y - \rho J^{-1}(Ay - f))\| \\
&\leq \|x - y - \rho J^{-1}A(x - y)\| \\
&\leq \|I - \rho J^{-1}A\| \|x - y\|.
\end{aligned}
$$

To evaluate the norm of $I - \rho J^{-1}A$, we have

$$\|(I - \rho J^{-1}A)x\|^2 = \|x\|^2 + \rho^2\|J^{-1}Ax\|^2 - 2\rho(x, J^{-1}Ax)$$

and

$$\|J^{-1}Ax\|^2 = \|Ax\| \leq \|A\| \|x\|.$$

Since A is H-elliptic,

$$(x, J^{-1}Ax) = (x, Ax) \geq c\|x\|^2.$$

Hence we conclude that

$$\|(I-\rho J^{-1}A)x\|^2 \leq (1+\rho^2\|A\|^2 - 2\rho c)\|x\|^2.$$

Then we choose $\rho = c\|A\|^{-2}$ and hence $L^2 = 1 - c\rho$. If $\|A\| = c$, then F is a constant and Fx is the solution of $(*)$ for any $x \in K$. If $\|A\| \neq c$, then the geometric estimations of the above theorem can be applied to this contraction F with the Lipschitzian constant $L<1$.

References

1. J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493-519.
2. Sehie Park, *Equivalent formulations of Ekeland's variational principle and their applications*, in "*Operator Equations and Fixed Point Theorems*" (ed. by S. P. Singh et al.), MSRI-Korea Publ. **1** (1986), 55-68.
3. T. E. Williamson, Jr., *Geometric estimation of fixed points of Lipschitzian mappings*, Bolletino U. M. I. **11** (1975), 536-543.

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