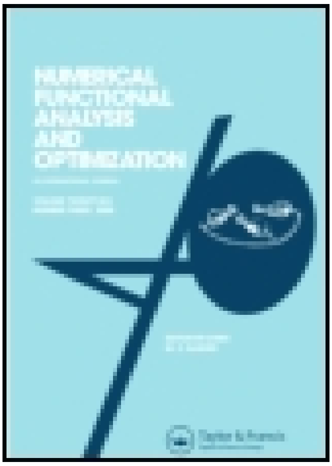

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On generalizations of ky fan's theorems on best approximations

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ON GENERALIZATIONS OF KY FAN'S THEOREMS ON BEST APPROXIMATIONS

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In 1969, Ky Fan [3] obtained the following existence result in best approximations :

THEOREM A (Ky Fan [3]). Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a $y_0 \in X$ such that $\|y_0 - fy_0\| \leq \|x - fy_0\|$ for all $x \in X$.

Since then there have appeared several extensions and applications of this theorem. For example, it has been applied to the proof of the Schauder fixed point theorem.

In the present paper, we give further generalizations of Ky Fan's theorem. In fact, our results include and strengthen those generalizations due to Ky Fan [3], [4], [5], Prolla [9], Park [8], and others. We also apply our results to obtain various generalizations of the Schauder fixed point theorem due to Ky Fan [3], Browder [1], Reich [10], and Halpern-Bergman [6].

We begin with the following :

THEOREM B (W.K.Kim [7]). Let X be a nonempty convex subset of a Hausdorff topological vector space E, and let $\phi : X \times X \to \mathbb{R}$ be a function. Suppose that

(a) $\phi(x,x) \leq 0$ for all $x \in X$,

(b) for each $x \in X$, the function $y \mapsto \phi(x,y)$ is quasi-concave on X,

(c) for each $y \in X$, the function $x \mapsto \phi(x,y)$ is l.s.c. on the intersection of X with any compact subset of E, and

(d) there exist a nonempty compact subset K of X and a nonempty set X_0, contained in some precompact convex subset of X, such that for each $x \in X \setminus K$, there exists a point $y \in X_0$ with $\phi(x,y) > 0$.

Then there exists an $x_0 \in K$ such that

$$\phi(x_0, y) \leq 0 \quad \text{for all} \quad y \in X.$$

Note that the condition (d) is fulfilled if X is compact and $X = K$. Theorem 1 implies a number of minimax theorems and variational inequalities as shown in [7].

Throughout this paper, Bd and Int denote the boundary and the interior in a normed vector space E. For any $X \subset E$, $\bar{I}_X(x)$ denotes the weakly inward set of X at $x \in E$, that is, the closure of the inward set

$$I_X(x) = \{x + r(u-x) \in E \mid u \in X, \quad r > 0\}.$$

Note that $X \subset I_X(x)$ and that if x is an internal point [2, p.410] of X, then $I_X(x) = E$. Note also that every interior point is an internal point [2, p.413].

We give the following application of Theorem B.

THEOREM 1.1. Let X be a nonempty convex subset of a normed vector space E, $g : X \to E$ and $f : X \to E$ continuous maps satisfying

$$\|gx - fx\| \leq \|x - fx\| \quad \text{for all } x \in X.$$

Let K be a nonempty compact subset of X and X_0 a nonempty subset of X contained in some precompact convex subset of X such that for each $x \in X \setminus K$, there exists a point $y \in X_0$ satisfying

$$\|gx - fx\| > \|y - fx\| .$$

Then there is an $x_0 \in K$ such that

$$\|gx_0 - fx_0\| \leq \|z - fx_0\| \quad \text{for all } z \in \bar{I}_X(gx_0).$$

More precisely, either

(a) f and g have a coincidence point x_0 in K, or

(b) there is an $x_0 \in K$ such that $gx_0 \in \text{Bd } K$ and

$$0 < \|gx_0 - fx_0\| \le \|z - fx_0\| \quad \text{for all} \quad z \in \overline{I}_X(gx_0).$$

Proof. Define $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \|gx - fx\| - \|y - fx\|.$$

Then the conditions (a), (b), (c), and (d) of Theorem B are clearly satisfied. Hence, there exists an $x_0 \in K$ such that

$$\|gx_0 - fx_0\| \le \|y - fx_0\| \quad \text{for all} \quad y \in X.$$

For $z \in I_X(gx_0) \setminus X$, there exist $u \in X$ and $r > 1$ such that $z = gx_0 + r(u - gx_0)$. Suppose that $\|gx_0 - fx_0\| > \|z - fx_0\|$. Since

$$u = \frac{1}{r} z + (1 - \frac{1}{r}) gx_0 \in X,$$

we have

$$\|u - fx_0\| \le \frac{1}{r} \|z - fx_0\| + (1 - \frac{1}{r}) \|gx_0 - fx_0\| < \|gx_0 - fx_0\|,$$

a contradiction. Therefore,

$$\|gx_0 - fx_0\| \le \|z - fx_0\| \quad \text{for all} \quad z \in \overline{I}_X(gx_0).$$

Now suppose that (a) does not hold. Then $\|gx_0 - fx_0\| > 0$. This implies $fx_0 \notin K \subset X \subset \overline{I}_X(gx_0)$. Suppose $gx_0 \in \operatorname{Int} K$. Then $fx_0 \notin \overline{I}_X(gx_0) = E$, a contradiction. Therefore, $gx_0 \in \operatorname{Bd} K$.

REMARKS. (1) If $g = 1_X$, Theorem 1.1 reduces to Theorem 1 of Park [8], which strengthens Theorem 7 of Ky Fan [5].

(2) If $g = 1_X$ and $K = X_0$, Theorem 1.1 reduces to Theorem 3 of Ky Fan [4], which generalizes his earlier result in [3].

The conclusion of Theorem 1.1 has some equivalent forms:

THEOREM 1.2. Under the hypothesis of Theorem 1.1, the following equivalent statements hold.

(i) There is an $x_0 \in K$ such that

$$\|gx_0-fx_0\| \leq \|z-fx_0\| \quad \text{for all} \quad z \in \bar{I}_X(gx_0).$$

(ii) If $h : X \to E$ is a map such that for any $x \in K$ with $gx \neq hx$, there exists a $y \in \bar{I}_X(gx)$ satisfying

$$\|gx - fx\| > \|y - fx\| ,$$

then h and g have a coincidence point in K.

(iii) If $h : X \to E$ is a map such that $hx \in \bar{I}_X(gx)$ for all $x \in X$ and

$$\|gx - fx\| > \|hx - fx\|$$

for all $x \in K$ with $gx \neq hx$, then h and g have a coincidence point in K.

Proof. (i) $\Rightarrow$ (ii) Suppose that $gx_0 \neq hx_0$. Then there exists a $y \in \bar{I}_X(gx_0)$ such that $\|gx_0-fx_0\| > \|y-fx_0\|$, a contradiction.

(ii) $\Rightarrow$ (iii) Put $y = hx$.

(iii) $\Rightarrow$ (i) Suppose that for any $x \in K$, there exists a $y \in \bar{I}_X(gx)$ such that $\|gx-fx\| > \|y-fx\|$. Note that, by the hypothesis of Theorem 1, for any $x \in X \setminus K$, there exists a point $y \in X_0 \subset X \subset I_X(x)$ satisfying $\|gx-fx\| > \|y-fx\|$. Choose hx to be one of such y. Then $h : X \to E$ satisfies $hx \in \bar{I}_X(x)$ for each $x \in X$. Note that h and g have no coincidence point by the definition. This contradicts (iii). Hence, there is an $x_0 \in K$ such that $\|gx_0-fx_0\| \leq \|z-fx_0\|$ for all $z \in \bar{I}_X(gx_0)$.

This completes our proof.

By putting $f = h$ is Theorem 1.2 we obtain the following:

THEOREM 1.3. Under the hypothesis of Theorem 1.1, f and g have a coincidence point in K if one of the following conditions holds :

(1) For any $x \in K$ with $fx \neq gx$ and $gx \in \mathrm{Bd}\, X$, there exists a $y \in \bar{I}_X(gx)$ satisfying $\|gx-fx\| > \|y-fx\|$.

(2) For any $x \in K$ with $fx \neq gx$ and $gx \in \mathrm{Bd}\, X$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that

$$|\lambda| < 1 \quad \text{and} \quad y := \lambda gx + (1-\lambda) fx \in \bar{I}_X(gx).$$

(3) For any $x \in K$ with $gx \in \mathrm{Bd}\, X$, we have $fx \in \bar{I}_X(gx)$.

Proof. (1) For any $x \in K$ with $fx \neq gx$ and $gx \in \mathrm{Int}\, X$, the point $y := (gx+fx)/2 \in \bar{I}_X(gx) = E$ satisfies $\|gx-fx\| > \|y-fx\|$. Therefore, by Theorem 1.2(ii), f and g have a coincidence point.

(2) We show that (2) $\Longrightarrow$ (1). In fact, we have

$$\|y-fx\| = |\lambda| \, \|gx-fx\| < \|gx-fx\| .$$

(3) For any $x \in K$ with $gx \in \operatorname{Int} X$, we clearly have $fx \in \bar{I}_X(gx) = E$. Therefore, by Theorem 1.2(iii) with $f = h$, f and g have a coincidence point.

REMARKS. (1) For $g = 1_X$, Theorem 1.3 reduces Corollaries 1.1 $\sim$ 1.4 in Park [8].

(2) For $g = 1_X$ and $X = K$, Theorem 1.3 includes earlier results of Browder [1], Ky Fan [3], and Halpern-Bergman [6].

For $X = K$, Theorems 1.1 $\sim$ 1.3 have stronger forms.

Let X be a convex subset of a normed space E and $g : X \to E$. The map g is said to be <u>almost</u> <u>affine</u> if

$$\|gx-y\| \leq \lambda \|gx_1-y\| + (1-\lambda) \|gx_2-y\|$$

for all $x_1, x_2 \in X$, $0 < \lambda < 1$, $x = \lambda x_1 + (1-\lambda)x_2$, and $y \in E$ [9].

A part of the following is motivated from J.B.Prolla [9].

THEOREM 2.1. Let X be a nonempty compact convex set in a normed space E, $g : X \to E$ and $f : X \to E$ continuous maps. If either

(a) $\|gx-fx\| \leq \|x-fx\|$ for all $x \in X$,

or

(b) $gX = X$ and g is almost affine, then there is an

$x_0 \in X$ such that

$$\|gx_0-fx_0\| \leq \|z-fx_0\| \quad \text{for all} \quad z \in \overline{I}_X(gx_0).$$

Proof. In case (a), the conclusion follows from Theorem 1.1 with $X = K$. In case (b), by the main result of Prolla [9], there is an $x_0 \in X$ such that $\|gx_0-fx_0\| \leq \|z-fx_0\|$ for all $z \in X$. Now, by the method of the proof of Theorem 1.1, the conclusion follows.

REMARKS. (1) As in Theorem 1.1, if $fx_0 \in \overline{I}_X(gx_0)$, then f and g have a coincidence point.

(2) If $g = 1_X$, Theorem 2.1 strengthens Theorem A of Ky Fan [3].

Imitating Theorems 1.2 and 1.3, we obtain the following :

THEOREM 2.2. Under the hypothesis of Theorem 2.1, the equivalent conditions (i), (ii), and (iii) of Theorem 1.2 with $X = K$ hold.

THEOREM 2.3. Under the hypothesis of Theorem 2.1, f and g have a coincidence point in X if one of the conditions (1), (2), and (3) of Theorem 1.3 with $X = K$ holds.

REMARKS. (1) For $g = 1_X$, Theorem 2.3 includes Corollaries 2.1 $\sim$ 2.4 in Park [8].

(2) For $g = 1_X$, Theorem 2.3 includes earlier genera-

lizations of the Schauder fixed point theorem due to Browder [1], Ky Fan [3], Reich [10], and Halpern-Bergman [6].

Finally, Theorems 1.1 and 2.1 with $g = 1_X$ imply the following approximation result.

COROLLARY. Under the hypothesis of Theorem 1.1 or 2.1 with $g = 1_X$, for any $z \in E$, there exists a $u \in K$ such that $||u-z|| = \min_{y \in K} ||y-z||$.

Proof. Let $f : X \to E$ be defined by the constant map $fx = z$ for all $x \in X$.

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