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# COMMON FIXED POINTS OF MAPS ON TOPOLOGICAL VECTOR SPACES HAVING SUFFICIENTLY MANY LINEAR FUNCTIONALS

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Fixed point theorems for upper semicontinuous (u. s. c.) multimaps on a nonempty compact convex subset of various topological vector spaces (t. v. s.) were obtained by S. Kakutani [9], Bohnenblust and Karlin [3], Ky Fan [11], Glicksberg [6], and others. Recently, W. K. Kim [10] and S. Park [14] generalized those results for a t. v. s. having sufficiently many linear functionals.

On the other hand, Itoh and Takahashi [7] proved a common fixed point theorem for a continuous map and an u. s. c. multimap on a compact convex subset of a locally convex space (l. c. s.) under some additional conditions.

In the present paper, we generalize their theorem for a t. v. s. having sufficiently many linear functionals, and also obtain a generalized version of the classical Markov-Kakutani theorem.

Let $E$ be a Hausdorff t. v. s. and $E^*$ its topological dual. $E$ is said to have sufficiently many linear functionals if for every $x \in E$ with $x \neq 0$ there exists a continuous linear functional $l \in E^*$ such that $l(x) \neq 0$. By the Hahn-Banach theorem, every l. c. s. has sufficiently many linear functionals. An example of a t. v. s. having sufficiently many linear functionals which is not locally convex is the Hardy space $H^p$ with $0 < p < 1$.

The following is a consequence of results in [14, 1, 2].

THEOREM 1. *Let $K$ be a nonempty compact convex subset of a Hausdorff t. v. s. $E$ having sufficiently many linear functionals, and $F: K \to 2^K$ a map such that $Fx$ is nonempty, closed, and convex for each $x \in K$. Then $F$ has a fixed point if one of the following holds:*

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(i) *F is continuous (u. s. c. and l. s. c.)*
(ii) *E is real and F is u. s. c.*
(iii) *E is locally convex and F is u. s. c.*
(iv) *E is real, locally convex and F is upper hemicontinuous.*

In case where $F$ is a single-valued map $f : K \to K$, Theorem 1 reduces to Ky Fan's theorem in [12]. Note that Theorem 1 includes results of Brouwer [4], Schauder [15], Tychonoff [16], Kakutani [9], Bohnenblust and Karlin [3], Glicksberg [6], Ky Fan [11, 12], and others.

By Theorem 1, the set $A=\mathrm{Fix}(f)=\{x\in K \mid x=fx\}$ is nonempty compact if $f : K \to K$ is continuous, and the set $B=\mathrm{Fix}(F)=\{x\in K \mid x\in Fx\}$ is nonempty compact if one of (i), (ii), and (iii) holds.

We say that $f$ and $F$ commute [7] if for each $x\in K$,

$$f(Fx)\subset F(fx).$$

THEOREM 2. *Under the hypothesis* (i), (ii), *or* (iii) *of* Theorem 1, *if $f : K \to K$ is continuous, $f$ and $F$ commute, and $A=\mathrm{Fix}(f)$ or $B=\mathrm{Fix}(F)$ is convex, then $f$ and $F$ have a common fixed point $z\in K$, that is, $z=fz\in Fz$.*

*Proof.* Suppose that $A$ is convex. Since $f(Fx)\subset F(fx)=Fx$ for each $x\in A$, $f$ is a continuous selfmap of a nonempty compact convex subset $Fx$ of $E$. Therefore, by Theorem 1, there is a $y\in Fx$ such that $y=fy$. Hence, $Fx\cap A$ is nonempty. Define a multimap $G : A\to 2^A$ by $Gx=Fx\cap A$ for $x\in A$. If $F$ is continuous [resp. u. s. c.], then $G$ is continuous [resp. u. s. c.] on the nonempty compact convex subset $A$ of $E$ and $Gx$ is nonempty closed convex for each $x\in A$. Thus, by Theorem 1, there exists a fixed point $z$ of $G$ in A. For this $z$, we have $z=fz\in Fz$.

Suppose that $B$ is convex. For any $x\in B$, we have $fx\in f(Fx)\subset F(fx)$. Hence, $f$ is a continuous selfmap of the nonempty compact convex subset $B$ of $E$. Therefore, by Theorem 1, there exists a point $z\in B$ such that $z=fz\in Fz$. This completes our proof.

For $f=1_K$, Theorem 2 reduces to Theorem 1, and for $F=1_K$, Theorem 2 reduces to Ky Fan's theorem in [12].

A map $F : K\to 2^K$ is said to be affine [7] if for any $x, y\in K$ and $\alpha\in[0,1]$,

$$\alpha Fx+(1-\alpha)Fy \subset F(\alpha x+(1-\alpha)y).$$

COROLLARY 1. *Under the hypothesis* (i), (ii) *or* (iii) *of* Theorem 1, *if* $f : K \to K$ *is continuous and affine, and* $f$ *and* $F$ *commute, then* $f$ *and* $F$ *have a common fixed point.*

*Proof.* Since $f$ is affine, the set $A$ is convex.

For $f = 1_K$, Corollary 1 reduces to Theorem 1.

COROLLARY 2. *Under the hypothesis* (i), (ii), *or* (iii) *of* Theorem 1, *if* $f : K \to K$ *is continuous,* $f$ *and* $F$ *commute, and* $F$ *is affine, then* $f$ *and* $F$ *have a common fixed point.*

*Proof.* Since $F$ is affine, the set $B$ is convex.

For $F = 1_K$, Corollary 2 redulces to Ky Fan's theorem in [12].

In [7], Itoh and Takahashi proved Theorem 2 and Corollaries 1 and 2 for locally convex $E$. Our proofs are slight modifications of theirs.

As an application of Theorem 2, we give the Markov-Kakutani theorem for a Hausdorff t. v. s. having sufficiently many linear functionals.

THEOREM 3. *Let* $K$ *be a nonempty compact convex subset of a Hausdorff t. v. s.* $E$ *having sufficiently many linear functionals. Let* $\mathfrak{F}$ *be a commuting family of continuous affine selfmaps of* $K$. *Then* $\mathfrak{F}$ *has a common fixed point.*

*Proof.* From Corollary 1, we know that for any $f, g \in \mathfrak{F}$, $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is nonempty compact convex. Hence so is any finite intersection of sets $\mathrm{Fix}(f)$, $f \in \mathfrak{F}$. Since $K$ is compact, the intersection of all sets $\mathrm{Fix}(f)$ is nonempty.

Theorem 3 for locally convex spaces was first given by Markov [13] with the aid of the Tychonoff fixed point theorem [16]. Kakutani [8] found a direct elementary proof of Theorem 3 (valid in any t. v. s.), and demonstrated the importance of the result by giving a number of applications; he also showed that Theorem 3 implies the Hahn-Banach theorem (see [5]). Our proof of Theorem 3 uses Ky Fan's theorem in [12] (i. e., the single-valued case of Theorem 1).

COROLLARY 3. *Let* $\mathfrak{F}$ *be the same in* Theorem 3 *and* $g : K \to K$ *a continuous map. If* $g$ *commutes with any* $f \in \mathfrak{F}$, *then* $\mathfrak{F} \cup \{g\}$ *has a common fixed point.*

*Proof.* The set $\mathrm{Fix}(\mathcal{F})$ of all common fixed points of $\mathcal{F}$ is nonempty compact convex by Theorem 3. Since $gz=gfz=fgz$ for any $z\in\mathrm{Fix}(\mathcal{F})$ and $f\in\mathcal{F}$, $g$ maps $\mathrm{Fix}(\mathcal{F})$ into itself. Thus $g$ has a fixed point in $\mathrm{Fix}(\mathcal{F})$ by Theorem 1. This completes our proof.

In fact, Corollary 3 shows that in Theorem 3 we can allow one map $f_0\in\mathcal{F}$ to be non-affine; there will still be a common fixed point for all maps $f$ in $\mathcal{F}$.

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