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Abstract

We obtain characterizations of metric completeness using certain partial orders and, by applying them, we improve various formulations of Ekeland's celebrated variational principle given by Caristi-Kirk-Browder, Kasahara, Siegel, Park, and Dancs-Hegedüs-Medvegyev.

In this paper, we obtain characterizations of metric completeness using certain partial orders, and then, by applying them, we improve various formulations of Ekeland's celebrated variational principle for approximate solutions of minimization problems. Consequently, previous results of Caristi-Kirk-Browder [5], Kasahara [4], Siegel [11], Park [8], and Dancs-Hegedüs-Medvegyev [1] are unified and generalized.

Let $\{F_\alpha\}_{\alpha\in I}$ be a family of nonempty subsets of a set X, where I is a simply ordered set, such that

$$\alpha\leq\beta \text{ in } I \quad \text{iff} \quad F_\beta\subset F_\alpha.$$

Then we can define a partial order $\leq$ on X corresponding to $\{F_\alpha\}$ by

$$x\leq y \text{ in } X \quad \text{iff} \quad x=y \text{ or there exists an } \alpha\in I \text{ such that } x\notin F_\alpha \text{ and } y\in F_\alpha.$$

LEMMA. $(X,\leq)$ *is a poset.*

Proof. Reflexivity is clear. For antisymmetry, suppose that $x\leq y$, $y\leq x$, and $x\neq y$. Thus $x\notin F_\alpha$, $y\in F_\alpha$, and $x\in F_\beta$, $y\notin F_\beta$ for some $\alpha,\beta\in I$. Therefore, either $F_\alpha\subset F_\beta$ or $F_\beta\subset F_\alpha$ leads a contradiction. For transitivity, suppose that $x\notin F_\alpha$, $y\in F_\alpha$, and $y\notin F_\beta$, $z\in F_\beta$ for some $\alpha,\beta\in I$. Then we should have $F_\beta\subset F_\alpha$, and hence $x\notin F_\beta$, $z\in F_\beta$. Therefore,

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$x<y$ and $y<z$ imply $x<z$.

By a *Cantor sequence* we mean a sequence $\{F_i\}_{i=1}^{\infty}$ of nonempty closed subsets of a metric space (X,d) such that

$$F_i \subset F_{i-1} \quad \text{and diam } F_i \to 0 \text{ as } i\to\infty.$$

The partial order $\le$ of X corresponding to a Cantor sequence will be called a *Cantor order*.

Now we have our characterizations of metric completeness.

THEOREM 1. *For any metric space (X,d), the following are equivalent:*

(i) *(X,d) is complete.*

(ii) *For any Cantor sequence $\{F_i\}$ of X, $\bigcap_{i=1}^{\infty} F_i$ is a singleton $\{p\}$ for some $p\in X$.*

(iii) *For any Cantor order $\le$ of X, it has the greatest element $p\in X$.*

(iv) *For any Cantor sequence $\{F_i\}$ of X, any sequence $\{x_i\}$ such that $x_i\in F_i$ has a unique upper bound $p\in X$ w.r.t. its Cantor order, $x_i\to p$, and p is independent of $\{x_i\}$.*

Proof. (i)$\Leftrightarrow$(ii) This is well-known. In fact, (i)$\Rightarrow$(ii) is due to G. Cantor, and (ii)$\Rightarrow$(i) to C. Kuratowski [6].

(ii)$\Rightarrow$(iii) For any $x\in X$, $x\neq p$, we have $x\in\bigcup_{i=1}^{\infty} F_i^c$. Then $x\notin F_i$ for some i. Since $p\in F_i$, we have $x<p$.

(iii)$\Rightarrow$(iv) Since p is the greatest, we have $x_i\le p$. Suppose that $x_i\le q$ for each i. Then $q\le p$. Suppose $q<p$. Then $q\notin F_j$ and $p\in F_j$ for some j. Then $q<x_j$, a contradiction. Therefore, p is the only upper bound of the nondecreasing sequence $\{x_i\}$.

(iv)$\Rightarrow$(ii) Choose a sequence $\{x_i\}$ such that $x_i\in F_i$ and let p be the upper bound of $\{x_i\}$. Suppose that $p\notin\bigcap_{i=1}^{\infty} F_i$. Then $p\notin F_i$ for some i. This implies $p<x_i$, a contradiction. Therefore $p\in\bigcap_{i=1}^{\infty} F_i$. Since diam $F_i\to 0$ as $i\to\infty$, $\bigcap_{i=1}^{\infty} F_i=\{p\}$. This also shows that $x_i\to p$ in (iv).

From Theorem 1, we obtain the following generalization of [1, Theorem 3.1].

THEOREM 2. *Let (X,d) be a metric space, $\Phi: X\to 2^X\setminus\{\phi\}$ a set-valued map, and $x_0\in X$ such that*

(1) *$\overline{\Phi(x_0)}$ is complete,*

(2) *for any $x,y\in X$, $y\in\overline{\Phi(x)}$ implies $\Phi(y)\subset\overline{\Phi(x)}$, and*

(3) *there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in X such that*

$$x_{i+1}\in\Phi(x_i) \quad \textit{and} \quad \text{diam } \Phi(x_i)\to 0 \textit{ as } i\to\infty.$$

Then Φ has a stationary point p in $\overline{\Phi(x_0)}$, that is, $\Phi(p)=\{p\}$, and $x_i\to p$.

Proof. Since $\overline{\Phi(x_{i+1})}\subset\overline{\Phi(x_i)}$ for each i, by Lemma, there exists a Cantor order $\leq$ on X corresponding to $\{\overline{\Phi(x_i)}\}$. Since $\overline{\Phi(x_0)}$ is complete, by Theorem 1, $\{x_i\}$ has a unique upper bound $p\in X$ such that $\bigcap_{i=0}^{\infty}\overline{\Phi(x_i)}=\{p\}$ and $x_i\to p$. Now it remains to show that p is a stationary point of Φ. Since $p\in\overline{\Phi(x_i)}$ for each i, $\Phi(p)\subset\overline{\Phi(x_i)}$ by (2). Therefore, $\Phi(p)\subset\bigcap_{i=0}^{\infty}\overline{\Phi(x_i)}=\{p\}$. Since $\Phi(p)\neq\phi$, we have $\Phi(p)=\{p\}$.

In [1, Theorem 3.1] the following condition is assumed instead of (3) in Theorem 2.

(3)′ $x\in\Phi(x)$ for each $x\in X$, and, for any sequence $\{x_i\}_{i=1}^{\infty}$ in X such that $x_{i+1}\in\Phi(x_i)$, we have $d(x_i, x_{i+1})\to 0$ as $i\to\infty$.

Clearly (3)′ implies (3) (cf. [1]).

Theorem 2 can also be proved by the results in [3]. In Theorems 1 and 2, we used only a part of the properties of a metric d. However, for convenience, we assumed that d is a metric.

Note that Theorems 3.1 and 3.2 in [1] are simple consequences of Theorem 2.

We now show that Siegel's main result in [11] is a consequence of Theorems 1 and 2.

Let X be a complete metric space and $F: X\to\boldsymbol{R}\cup\{+\infty\}$ a function, $\not\equiv+\infty$, bounded from below. Let $\mathscr{F}$ be the family of selfmaps f of X satisfying

$$F(fx)\leq F(x)-d(x,fx),\quad x\in X.$$

Note that $\mathscr{F}$ is closed under composition and that if F is l.s.c. then $\mathscr{F}$ is closed under countable composition [11].

THEOREM 3. [11] *Let $\mathscr{G}\subset\mathscr{F}$ be closed under composition. Let $x_0\in X$ such that $F(x_0)<+\infty$.*

(a) *If $\mathscr{G}$ is closed under countable composition, then there exists an $\bar{f}\in\mathscr{G}$ such that $\bar{x}=\bar{f}x_0$ and $g\bar{x}=\bar{x}$ for each $g\in\mathscr{G}$.*

(b) *If each $g\in\mathscr{G}$ is continuous, then there exist a sequence $f_i\in\mathscr{G}$ and a point $\bar{x}=\lim_{i\to\infty} f_i f_{i-1}\cdots f_1(x_0)$ such that $g\bar{x}=\bar{x}$ for each $g\in\mathscr{G}$.*

Proof. Without loss of generality, we may assume that the identity map 1_X belongs to $\mathscr{G}$. Let $\mathscr{G}(x)=\{gx\,|\,g\in\mathscr{G}\}$. We choose sequences $\{x_i\}_{i=0}^{\infty}$ in X and $\{f_i\}_{i=1}^{\infty}$ in $\mathscr{G}$ by induction as follows: If $\mathscr{G}(x_i)=\{x_i\}$, then set $x_{i+1}=x_i$ and let f_{i+1} be any map in $\mathscr{G}$. If $\mathscr{G}(x_i)\supsetneqq\{x_i\}$, then choose $f_{i+1}\in\mathscr{G}$ and set $x_{i+1}=f_{i+1}x_i\in\mathscr{G}(x_i)$ such that

$$d(x_i, x_{i+1})\geq(\text{diam }\mathscr{G}(x_i))/2-1/2^i.$$

Then we have

$$F(x_{i+1})\leq F(x_i)-d(x_i, x_{i+1}),$$

and $\{F(x_i)\}$ is nonincreasing. Since F is $\not\equiv+\infty$, bounded from below, and $F(x_0)<+\infty$, we have $d(x_i, x_{i+1})\to 0$ as $i\to\infty$. Therefore, diam $\mathscr{G}(x_i)\to 0$ as $i\to\infty$. Since $\mathscr{G}$ is closed under composition, we have $\mathscr{G}(x_{i+1})\subset\mathscr{G}(x_i)$. Since diam $\overline{\mathscr{G}(x_i)}\to 0$ as $i\to\infty$, by Theorem 1, $x_i\to\bar{x}$ for some $\bar{x}\in X$.

(a) Let $\bar{f}=\Pi_{i=1}^{\infty} f_i\in\mathscr{G}$. Then $\bar{x}=\bar{f}x_0$. Since $\bar{x}=(\Pi_{j=i+1}^{\infty} f_j)x_i$ for each i, we have $\bar{x}\in\mathscr{G}(x_i)$. Since diam $\mathscr{G}(x_i)\to 0$, we have $\{\bar{x}\}=\bigcap_{i=0}^{\infty}\mathscr{G}(x_i)$. It remains to show $g\bar{x}=\bar{x}$ for each $g\in\mathscr{G}$. Since $g\bar{x}=g(\Pi_{j=i+1}^{\infty} f_j)x_i\in\mathscr{G}(x_i)$ for each i, we must have $g\bar{x}=\bar{x}$.

(b) Let $\Phi(x)=\mathscr{G}(x)$ for each $x\in X$. Then $\Phi: X\to 2^X\setminus\{\phi\}$ satisfies the hypothesis of Theorem 2. In fact, (1) is clear. Since $\mathscr{G}$ is closed under composition and each $g\in\mathscr{G}$ is continuous, (2) also holds. For the sequence $\{x_i\}$, we have diam $\Phi(x_i)\to 0$ as $i\to\infty$. Therefore (3) holds. Now, by Theorem 2, Φ has a stationary point $\bar{x}\in\overline{\Phi(x_0)}$ such that $x_i\to\bar{x}$, that is, $\bar{x}=\lim_{i\to\infty} f_i f_{i-1}\cdots f_1(x_0)$. Since $g\bar{x}\in\Phi(\bar{x})=\{\bar{x}\}$ for each $g\in\mathscr{G}$, we must have $g\bar{x}=\bar{x}$.

In the proof of (b), the continuity of each map in $\mathscr{G}$ is needed only for the condition (2). For $\mathscr{G}=\{g^n\}$, the class consisting of g and its finite iterates, certain condition, e.g. so-called orbital continuity, on g suffices to guarantee the condition (2). For such choice of $\mathscr{G}$ one has $\bar{x}=\lim_{i\to\infty} g^i x_0$ as in the Banach contraction principle.

An improved version of Theorem 3 is the following:

THEOREM 4. *Let X be a metric space, and $F: X\to\boldsymbol{R}\cup\{+\infty\}$ a function, $\not\equiv+\infty$, bounded from below. Let $\varepsilon>0$ and $\lambda>0$ be given, and a point $u\in X$ such that $F(u)\leq\inf_X F+\varepsilon$. Let $A=\{x\in X\mid F(x)\leq F(u)-\varepsilon\lambda^{-1}d(u,x)\}$ be complete and $\mathscr{F}$ be the family of selfmaps f of X satisfying*

$$F(fx)\leq F(x)-\varepsilon\lambda^{-1}d(x, fx),\quad x\in X.$$

Let $\mathscr{G}\subset\mathscr{F}$ be closed under composition.

(a) *If $\mathscr{G}$ is closed under countable composition, then there exists an $\bar{f}\in\mathscr{G}$ such that $\bar{x}=\bar{f}u\in A$ and $g\bar{x}=\bar{x}$ for each $g\in\mathscr{G}$.*

(b) *If each $g\in\mathscr{G}$ is continuous, then there exist a sequence $f_i\in\mathscr{G}$ and a point $\bar{x}=\lim_{i\to\infty} f_i f_{i-1}\cdots f_1(u)\in A$ such that $g\bar{x}=\bar{x}$ for each $g\in\mathscr{G}$.*

Proof. Note that for any $f\in\mathscr{F}$, we have $fA\subset A$. In fact, for any $x\in A$,

$$F(x)\leq F(u)-\varepsilon\lambda^{-1}d(u,x)\quad\text{and}\quad F(fx)\leq F(x)-\varepsilon\lambda^{-1}d(x,fx)$$

imply

$$F(fx)\leq F(u)-\varepsilon\lambda^{-1}d(u,fx).$$

Now replacing X and d in Theorem 3 by A and $\varepsilon\lambda^{-1}d$, respectively, we obtain the consclusion.

Note that if X is complete and F is l.s.c. then A is closed and hence complete.

Also note that

$$A\subset\{x\in X\mid F(x)\leq F(u),\ d(u,x)\leq\lambda\}\subset\bar{B}(u,\lambda),$$

because for any $x\in A$ we have

$$\varepsilon\lambda^{-1}d(u,x)\leq F(u)-F(x)\leq F(u)-\inf_X F\leq\varepsilon.$$

Therefore, comparing with Theorem 3, Theorem 4 gives the geometric location of the common fixed point.

Moreover, Theorem 3 follows from Theorem 4. In fact, for any $x_0\in X$ with $F(x_0)<+\infty$, we can find an $\varepsilon>0$ such that $F(x_0)\leq\inf_X F+\varepsilon$. If $0<\varepsilon\leq 1$, then $F(fx)\leq F(x)-d(x,fx)$ implies $F(fx)\leq F(x)-\varepsilon d(x,fx)$ for each $f\in\mathcal{F}$. Therefore, Theorem 3 follows from Theorem 4 for $\lambda=1$ and $A=\{x\in X\mid F(x)\leq F(x_0)-\varepsilon d(x,x_0)\}$. If $\varepsilon<1$, then by choosing $F_1=F/\varepsilon$, we have $F_1(x_0)\leq\inf_X F_1+1$ and $F_1(fx)\leq F_1(x)-\varepsilon^{-1}d(x,fx)$ for $x\in X$ and $f\in\mathcal{F}$, which reduces to the first case.

Finally, we give simple proofs of Ekeland's variational principle. In order to do this, consider the following equivalent formulations of Ekeland's theorem in [8], whose equivalency is a simple consequence of a metatheorem in [9].

THEOREM 5. *Let X be a complete metric space, and $F: X\to\mathbf{R}\cup\{+\infty\}$ a l.s.c. function, $\not\equiv+\infty$, bounded from below. Let $\varepsilon>0$ be given, a point $u\in X$ such that $F(u)\leq\inf_X F+\varepsilon$, and $A=\{x\in X\mid F(x)\leq F(u)-\varepsilon d(u,x)\}$.*

Then the following equivalent statements hold:

(i) *There exists a point $v\in A$ such that*

$$\forall w\neq v,\ F(w)>F(v)-\varepsilon d(v,w).$$

(ii) If $T: A\to 2^X$ *satisfies the condition:*

$$\forall x\in A\setminus T(x)\ \exists y\in X\setminus\{x\}\ \textit{such that}$$
$$F(y)\leq F(x)-\varepsilon d(x,y),$$

then T has a fixed point $v\in A$, that is, $v\in T(v)$.

(iii) *If $f: A\to X$ is a map satisfying*

$$F(fx)\leq F(x)-\varepsilon d(x,fx)$$

for all $x\in A$, then f has a fixed point.

(iv) *If $T: \to 2^X\setminus\{\phi\}$ satisfies the condition:*

$$\forall x\in A \ \forall y\in T(x),\ F(y)\leq F(x)-\varepsilon d(x,y),$$

then T has a stationary point $v\in A$.

Proof. (iii) Note that $fA\subset A$. Let $\mathcal{F}$ be the family of selfmaps f of A satisfying $F(fx)\leq F(x)-\varepsilon d(x,fx)$, $x\in A$. Since F is l.s.c., A is complete and, as we noted earlier, $\mathcal{F}$ is closed under countable composition. Therefore, by Theorem 3(a) or 4(a), $\mathcal{F}$ has a common fixed point.

More directly, we can prove Theorem 5 from Theorem 2.

Proof. (iv) Define $\Phi: A\to 2^X$ by

$$\Phi(x)=\{y\in X\,|\,F(y)\leq F(x)-\varepsilon d(x,y)\}\subset A$$

for each $x\in A$. Then $\Phi(x)$ is nonempty and closed since F is l.s.c. and hence $A=\Phi(u)$ is closed. Therefore, (1) holds. If $y\in\Phi(x)$ and $z\in\Phi(y)$, then

$$F(y)\leq F(x)-\varepsilon d(x,y) \quad\text{and}\quad F(z)\leq F(y)-\varepsilon d(y,z)$$

imply

$$F(z)\leq F(x)-\varepsilon d(x,z).$$

Therefore, $z\in\Phi(x)$. Thus (2) holds. Choose $u=u_0$. Suppose $u_i\in A$ is known. If $\Phi(u_i)=\{u_i\}$, then set $u_{i+1}=u_i$. Otherwise, choose $u_{i+1}\in\Phi(u_i)$ such that

$$d(u_i,u_{i+1})\leq(\text{diam }\Phi(u_i))/2-1/2^i.$$

Then we have

$$F(u_{i+1})\leq F(u_i)-\varepsilon d(u_i,u_{i+1}),$$

and hence $d(u_i,u_{i+1})\to 0$ as $i\to\infty$. Thus diam $\Phi(u_i)\to 0$. Therefore, (3) holds. Thus, by Theorem 2, Φ has a stationary point $v\in\Phi(v)=A$. Now $\phi\neq T(v)\subset\Phi(v)=\{v\}$ implies $\{v\}=T(v)$, and $u_n\to v$.

In Theorem 5, if A is complete, we do not need to assume that X is complete.

Actually, Theorem 5(i) is a little stronger than Ekeland's variational principle [2], for

$$A\subset\{x\in X\,|\,F(x)\leq F(u),\ d(u,x)\leq 1\}\subset\bar{B}(u,1).$$

Theorem 5(iii) is known as the Caristi-Kirk-Browder fixed point theorem, and this characterizes metric completeness. Since Theorem 1 implies Theorems 2~5, we can conclude that Theorems 2~4 and each of Theorem 5 also characterize metric completeness (cf. [7, 10, 11, 12, 13]).

A simple consequence of Theorem 5(iv) or Proof (iii) is also obtained by S. Kasahara [4] for L-spaces.

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