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Seoul National University, Seoul, Korea

— Dedicated to Professor Han Shick Park on his 60th birthday —

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Equivalent Formulations of Zorn's Lemma and Other Maximum Principles

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Abstract. In this paper, we give a result that maximum principles including Zorn's lemma can be regarded as various types of fixed point theorems. Our main application is that the well-known ordering principles in nonlinear analysis including the Bishop-Phelps argument and a number of its generalizations can be converted to fixed point theorems and vice versa. Consequently, we obtain new results and unify many known results.

Recently, there have been appeared a number of ordering principles in nonlinear analysis including the well-known Bishop-Phelps lemma [4] and its extensions given by Phelps [22], Ekeland [13], Brøndsted [8], Brézis-Browder [7], Altman [3], Turinici [26], [28], and Kang-Park [15]. It is well-known that any maximum principle including Zorn's lemma implies a fixed point result on expanding maps f (that is, $x \leq fx$ for all x).

In this paper, we begin with a metatheorem on the equivalency of maximum principles and various types of fixed point results. We apply this to Zorn's lemma and some useful forms of the above-mentioned principles. Consequently, we obtain some new results and unify a number of known results.

Let 2^X denote the power set of a set X , and \sim the negation.

In the Zermelo-Fraenkel set theory with Axiom of Choice, we have the following.

Theorem 1. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following are equivalent:*

- (i) *There exists an element $v \in A$ such that $G(v, w)$ for any $w \in X \setminus \{v\}$.*
- (ii) *If $T : A \rightarrow 2^X$ is a multimap such that for any $x \in A \setminus T(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x, y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*
- (iii) *If $f : A \rightarrow X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x, y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*
- (iv) *If $T : A \rightarrow 2^X \setminus \{\emptyset\}$ is a multimap such that $\sim G(x, y)$ holds for any $x \in A$ and any $y \in T(x) \setminus \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

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(v) If \mathcal{F} is a family of maps $f: A \rightarrow X$ satisfying $\sim G(x, fx)$ for all $x \in A$ with $x \neq fx$, then \mathcal{F} has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathcal{F}$.

Proof. See [22].

A nonempty poset (partially ordered set) is said to be *inductively ordered* if

(A) every nonempty simply ordered subset has an upper bound.

We adopt the following form of Zorn's lemma.

Theorem. Let (P, \leq) be an inductively ordered set. Then for any $a \in P$, there exists a maximal element $v \in P$ such that $v \in S(a) = \{x \in P \mid a \leq x\}$.

An equivalent form of Zorn's lemma is obtained by replacing (A) by the following ([2], [5], [6]):

(B) every nonempty well-ordered subset has an upper bound.

As the first application of Theorem 1, we have the following:

Theorem 2. Let (P, \leq) be a poset, $a \in P$, and $S(a) = \{x \in P \mid a \leq x\}$ satisfy (A) or (B). Then we have the following equivalent conditions hold:

(i) There exists a $v \in S(a)$ such that $\sim(v \leq w)$ for any $w \in P \setminus \{v\}$.

(ii) If $T: S(a) \rightarrow 2^P$ is a multimap such that

$$\forall x \in S(a) \setminus T(x), \exists y \in P \setminus \{x\} \text{ such that } x \leq y,$$

then T has a fixed element $v \in S(a)$.

(iii) If $f: S(a) \rightarrow P$ is a map such that $x \leq fx$ for any $x \in S(a)$, then f has a fixed element $v \in S(a)$.

(iv) If $T: S(a) \rightarrow 2^P \setminus \{\emptyset\}$ is a multimap such that

$$\forall x \in S(a), \forall y \in T(x), x \leq y \text{ holds,}$$

then T has a stationary element $v \in S(a)$.

(v) If \mathcal{F} is a family of maps $f: S(a) \rightarrow P$ satisfying $x \leq fx$ for all $x \in S(a)$, then \mathcal{F} has a common fixed element $v \in S(a)$.

Note that (i) is equivalent to Zorn's lemma and (iii) due to Abian [1], Kneser [17] and Moroi-anu [19] with constructive proofs.

Actually, (i) \implies (iii) is a simple observation ([9]). (iii) \implies (i) is given in [1], [2], [19]. Kasahara [16] obtained (i) \implies (v). Also note that Proposition 1.6 of [24] is a consequence of (i) \implies (iii).

A slightly different version of Theorem 2 can be stated as follows:

Theorem 3. Let (P, \leq) be a poset and $A \subset P$ a nonempty subset. Then the followings are equivalent.

(i) A has a maximal (minimal) element.

(ii) If $T: A \rightarrow 2^P$ is a multimap such that

$$\forall x \in A \setminus T(x), \exists y \in A \text{ such that } x < y (y < x),$$

then T has a fixed element.

(iii) If $f: A \rightarrow P$ is a map such that

$$\forall x \in A, x \neq fx \implies x < fx (fx < x) \text{ and } fx \in A,$$

then f has a fixed element.

(iv) If $T: A \rightarrow 2^P \setminus \{\emptyset\}$ is a multimap such that

$$\forall x \in A, \forall y \in T(x) \setminus \{x\}, x < y (y < x) \text{ and } y \in A,$$

then T has a stationary element.

(v) If \mathcal{F} is a family of maps $f: A \rightarrow P$ satisfying

$$\forall x \in A, x \neq fx \implies x < fx (fx < x) \text{ and } fx \in A,$$

then \mathcal{F} has a common fixed element.

Let (P, \leq) be a poset and $f: P \rightarrow P$ a selfmap. Then f is said to be *isotone* if $f(x) \leq f(y)$ whenever $x \leq y$.

Corollary 1. (Smithson [25]). *Let (P, \leq) be a poset. Let $e \in P$ and \mathcal{F} be a commuting family of isotone maps of P (i.e., $f \circ g \cong g \circ f$ for all $f, g \in \mathcal{F}$). Suppose that $e \leq fe$ for all $f \in \mathcal{F}$ and every chain containing e has a least upper bound in P , then \mathcal{F} has a common fixed element.*

Proof. Let $A = \{x \in P \mid e \leq x \text{ and } x \leq fx \text{ for all } f \in \mathcal{F}\}$. Then $e \in A$ and $A \neq \emptyset$. Let C be a chain in A and $x_0 = \text{lub } C$. Then for any $f \in \mathcal{F}$ and any $x \in C$, $x \leq x_0$ implies $x \leq fx \leq fx_0$. So fx_0 is an upper bound for C , whence $x_0 \leq fx_0$. That is, $x_0 \in A$. By Zorn's lemma, A has a maximal element.

Let $x \in A$ and $f \in \mathcal{F}$. If $x \neq fx$, $e \in x \setminus fx$. And for each $g \in \mathcal{F}$, $x < gx$ implies $g(fx) = f(gx) \leq f(x)$. Thus $fx \in A$. By (v) of Theorem 3, \mathcal{F} has a common fixed element.

Corollary 1 generalizes Knaster-Tarski's theorem (Theorem I.4.1. in [11]), which also can be derived from (i) \iff (iii) of Theorem 3. Also note that De Marr's theorem [11] is a dual form of Corollary 1.

Corollary 2. (Höft and Höft [14]). *Let (P, \leq) be a poset. Suppose that every nonempty chain in P has a l.u.b. and a g.l.b. in P . If $f: P \rightarrow P$ is an isotone map and there is an $e \in P$ such that e and fe are comparable, then f has a fixed element.*

Proof. This is an easy consequence of (i) \iff (iii) of Theorem 3, by setting

$$A = \{x \in P \mid x \leq fx \text{ and } e \leq x\} \text{ if } e \leq fe \text{ or setting } A = \{x \in P \mid fx \leq x \text{ and } x \leq e\} \text{ if } fe \leq e.$$

A number of earlier fixed point results on posets may follow from Theorem 2 or Theorem 3.

In nonlinear analysis, there have appeared a number of constructive maximum principles. In fact, a set (X, \leq) with a quasi-order \leq (that is, reflexive and transitive) has some additional structure like a metric space, we have the following application of Theorem 1, which can be regarded a constructive "countably inductive" version of Zorn's lemma.

Theorem 4. *Let (X, d, \leq) be a quasi-ordered metric space, and $a \in X$ such that*

- (1) *any nondecreasing Cauchy sequence in $S(a)$ has an upper bound, and*
- (2) *for any $x \in S(a)$ and $\varepsilon > 0$, there exists $y \in S(x)$ such that $\text{diam } S(y) < \varepsilon$.*

Then (i)–(v) of Theorem 2 hold.

Proof. (i) By (2), since $a \in S(a)$, there exists a $y_1 \in S(a)$ such that $\text{diam } S(y_1) < 1$. Suppose $y_1, \dots, y_n \in S(a)$ are chosen. Then there exists $y_{n+1} \in S(y_n)$ such that $\text{diam } S(y_{n+1}) < 1/(n+1)$. By induction, we obtain a nondecreasing Cauchy sequence $\{y_n\}$. By (1), there exists a $v \in S(a)$ such that $y_n \leq v$ for all n . Since $v \in S(y_n)$ and $\text{diam } S(y_n) \rightarrow 0$ as $n \rightarrow \infty$, we have $y_n \rightarrow v$. Since $v \leq x$ implies $v, z \in S(y_n)$, we have $d(v, z) < 1/n$ for all n , and hence $v = z$.

Note that an upper bound in (1) is actually a maximal element.

Theorem 4(i) is due to Turinici [26], [27], [28] in a more general form. For far reaching generalization of Theorem 4(i) is obtained recently by Kang and Park [15]. The essential features of those generalizations imply well-known ordering principles of Altman [3], Brézis-Browder [7], Brøndsted [8], and many others. Those principles unify a number of diverse results in nonlinear analysis. Note also that (i) \implies (iii) is given by Turinici [26].

The condition (2) is implied by

(2)' any nondecreasing sequence $\{x_n\}$ is *d-asymptotic* (that is, $\liminf_n d(x_n, x_{n+1}) = 0$).

For complete metric spaces, the following types of maximum principles have been widely used, e.g. the Banach contraction principle, the Bishop-Phelps lemma [4], and many of their extensions.

Theorem 5. Let (X_0, d) be a metric space, $\phi : X_0 \rightarrow \{-\infty\} \cup \mathbf{R}$ u.s.c., bounded above, and $k > 0$. Define a partial order \leq on $X = \{x \in X_0 \mid \phi(x) > -\infty\}$ by

$$x \leq y \text{ iff } kd(x, y) \leq \phi(y) - \phi(x).$$

Let $a \in X$, and suppose $S(a)$ is \leq -complete (that is, every nondecreasing Cauchy sequence converges).

Then (i)–(v) of Theorem 2 hold.

Proof. (i) We claim that any nondecreasing sequence $\{x_n\}$ is Cauchy. In fact, if $m \geq n$, then $kd(x_n, x_m) \leq \phi(x_m) - \phi(x_n)$ implies $\phi(x_m) \geq \phi(x_n)$. Since $\phi(x_n)$ is nondecreasing and ϕ is bounded above, $\phi(x_n) \uparrow c$ for some $c \in \mathbf{R}$. This shows that $\{x_n\}$ is Cauchy and hence satisfies (2)'. Since $S(a)$ is \leq -complete, if $\{x_n\}$ is in $S(a)$, then $x_n \rightarrow x$ for some $x \in S(a)$. In fact, $x \in X$, for $\phi(x) \geq \limsup_n \phi(x_n) = c$ since ϕ is u.s.c. Now we claim that x is an upper bound of $\{x_n\}$. In fact,

$$\begin{aligned} kd(x, x_n) &= \lim_{m \geq n} kd(x_m, x_n) \\ &\leq \limsup_{m \geq n} \phi(x_m) - \phi(x_n) \\ &= a - \phi(x_n) \leq \phi(x) - \phi(x_n). \end{aligned}$$

Therefore, by Theorem 3, we have the conclusion.

In Theorem 5, since ϕ is u.s.c., $S(a)$ is closed. Therefore, if (X_0, d) is complete, then clearly $S(a)$ is complete and hence \leq -complete. Moreover, if we choose $a \in X$ such that $\phi(a) \leq \sup_X \phi - k$, then $S(a) = \{x \in X \mid kd(x, a) \leq \phi(x) - \phi(a)\} \subset \{x \in X \mid \phi(x) > \phi(a), d(x, a) \leq 1\}$. This gives more accurate informations on whereabouts of locations of maximal points or fixed points.

Theorem 5(i) is due to Phelps [23] and extends the well-known Bishop-Phelps argument in [4]. Actually, Phelps proved (i) by using Zorn's lemma.

A dual form of Theorem 5 can be stated as follows:

Theorem 6. Let (X_0, d) be a metric space, $\phi : X_0 \rightarrow \mathbf{R} \cup \{+\infty\}$ l.s.c., bounded below, and $k > 0$. Define a partial order \leq on $X = \{x \in X_0 \mid \phi(x) < +\infty\}$ by

$$x \leq y \text{ iff } kd(x, y) \leq \phi(x) - \phi(y).$$

Let $a \in X$, and suppose $S(a)$ is \leq -complete. Then (i)–(v) of Theorem 2 hold.

Proof. Take $-\phi$ instead of ϕ and apply Theorem 5.

In Theorem 6, since ϕ is l.s.c., $S(a)$ is closed. Therefore, if (X_0, d) is complete, then clearly $S(a)$ is complete and hence \leq -complete. Moreover, if we choose $a \in X$ such that $\phi(a) \leq \inf_X \phi + k$,

then $S(a) = \{x \in X \mid kd(x, a) \leq \phi(a) - \phi(x)\} \subset \{x \in X \mid \phi(x) < \phi(a), d(x, a) \leq 1\}$.

Theorem 6 is given in Park [20], [21]. Actually, Theorem 6(i) is the celebrated variational principle of Ekeland [13], (ii) essentially due to Tuy [29], (v) to Kasahara [16], (iv) to Maschler-Peleg [18], and (iii) to Caristi-Kirk-Browder [10], which includes the Banach contraction principle. Applications of Theorem 6 are numerous in a vast field of mathematical sciences (see, e.g., [13], [20], [21]).

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