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FIXED POINTS OF ROTATIVE LIPSCHITZIAN MAPS

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1. Introduction

Let X be a closed convex subset of a Banach space B and $T: X \longrightarrow X$ a lipschitzian rotative map, i.e., such that $\|Tx-Ty\| \leq k\|x-y\|$ and $\|T^n x - x\| \leq a\|Tx-x\|$ for some real k, a and an integer $n > a$. We denote by $\Phi(n, a, k, X)$ the family of all such maps.

In [3], [4], K. Goebel and M. Koter obtained results concerning the existence of fixed points of T depending on k, a and n.

In the present paper, the main results of [3], [4] are so strengthened that some information concerning the geometric estimations of fixed points are given.

2. Preliminaries

Our tool in this paper is the following in [5], which is a consequence of the well-known variational principle of Ekeland [1], [2] for approximate solutions of minimization problems.

THEOREM 0. *Let V be a complete metric space and $f: V \longrightarrow V$ be a map such that there exists an $L \in [0, 1]$ satisfying*

$$d(fx, f^2 x) \leq L d(x, fx) \quad \text{for any } x \in V.$$

If $F(x) = d(x, fx)$ on V is l.s.c., then

(1) $\lim f^n x = p$ *exists for any* $x \in V$,

$$d(f^n x, p) \leq \frac{L^n}{1-L} d(x, fx)$$

and p is a fixed point of f, and

(2) *for any $u \in V$ and $\varepsilon > 0$ satisfying*

$$F(u) \leq (1-L)\varepsilon$$

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f has a fixed point in $\overline{B}(u, \varepsilon)$. Further, if f is a quasi-lipschitzian with constant k, then either u is a fixed point of f or f has a fixed point in

$$\overline{B}(u, \varepsilon) \backslash B(u, s)$$

where $s = d(u, fu)(1+k)^{-1}$.

3. Main results

Note that if T is k-lipschitzian, then $I-\alpha T$ is invertible for $\alpha \in (0, 1/k)$. Thus we can consider the map $T_\alpha : X \longrightarrow X$ defined by

$$T_\alpha = (I-\alpha T)^{-1}(1-\alpha)I \quad \text{or} \quad T_\alpha x = (1-\alpha)x + \alpha T T_\alpha x.$$

It is easy to see that Fix T_α=Fix T, and for a k-lipschitzian map T, T_α is a $(1-\alpha)(1-\alpha k)^{-1}$-lipschitzian map.

THEOREM 1. *If $T \in \Phi(n, a, 1, X)$ for some $n \in N$ and $a \in [0, n)$, then for any $\alpha \in (0, 1)$ such that*

$$g(\alpha) := (a+n)\left(\textstyle\sum_{i=0}^{n-1} \alpha^i\right)^{-1} - 1 < 1, \quad u \in X \text{ and } \varepsilon > 0$$

satisfying

$$\|u - T_\alpha u\| \leq (1-g(\alpha))\varepsilon,$$

either u is a fixed point of T or there is a fixed point of T in $\overline{B}(u, \varepsilon) \cap X \backslash B(u, s)$ where $s = \|u - T_\alpha u\|/2$.

Proof. We follow the method in [4].

From $\quad T_\alpha x = (1-\alpha)x + \alpha T T_\alpha x$

we have $\quad T_\alpha^2 x = (1-\alpha) T_\alpha x + \alpha T T_\alpha^2 x, \quad \cdots$etc.

and $\quad (1-\alpha)(x - T_\alpha x) = \alpha(T_\alpha x - T T_\alpha x)$.

Thus we have

$$\begin{aligned}
\|T_\alpha x - T_\alpha^2 x\| &= \|T_\alpha x - (1-\alpha) T_\alpha x - \alpha T T_\alpha^2 x\| \\
&= \alpha \|T_\alpha x - T T_\alpha^2 x\| \\
&\leq \alpha \|T_\alpha x - T^n T_\alpha x\| + \alpha \|T^n T_\alpha x - T T_\alpha^2 x\| \\
&\leq \alpha a \|T_\alpha x - T T_\alpha x\| + \alpha \|T^{n-1} T_\alpha x - T_\alpha^2 x\| \\
&= (1-\alpha) a \|x - T_\alpha x\| + \alpha \|T^{n-1} T_\alpha x - T_\alpha^2 x\|.
\end{aligned}$$

Using only the nonexpansiveness of T, we proceed by induction to establish the following inequality which is needed:

$$\alpha \|T^{k-1} T_\alpha x - T_\alpha^2 x\| \leq \{(k-1) - k\alpha + \alpha^k\} \|x - T_\alpha x\| + \alpha^k \|T_\alpha x - T_\alpha^2 x\|.$$

For $k=2$,

$$\begin{aligned}\alpha\|TT_\alpha x-T_\alpha{}^2x\| &\le \alpha\|TT_\alpha x-(1-\alpha)T_\alpha x-\alpha TT_\alpha{}^2x-\alpha TT_\alpha x+\alpha TT_\alpha x\|\\ &\le \alpha(1-\alpha)\|TT_\alpha x-T_\alpha x\|+\alpha^2\|TT_\alpha x-TT_\alpha{}^2x\|\\ &\le (1-\alpha)^2\|x-T_\alpha x\|+\alpha^2\|T_\alpha x-T_\alpha{}^2x\|.\end{aligned}$$

For $k=n+1$,

$$\begin{aligned}\alpha\|T^nT_\alpha x-T_\alpha{}^2x\| &= \alpha\|(1-\alpha)T^nT_\alpha x+\alpha T^nT_\alpha x-(1-\alpha)T_\alpha x-\alpha TT_\alpha{}^2x\|\\ &\le \alpha(1-\alpha)\|T^nT_\alpha x-T_\alpha x\|+\alpha^2\|T^nT_\alpha x-TT_\alpha{}^2x\|\\ &\le n\alpha(1-\alpha)\|TT_\alpha x-T_\alpha x\|+\alpha^2\|T^{n-1}T_\alpha x-T_\alpha{}^2x\|\\ &= n(1-\alpha)^2\|x-T_\alpha x\|+\alpha^2\|T^{n-1}T_\alpha x-T_\alpha{}^2x\|\end{aligned}$$

and by the induction hypothesis

$$\begin{aligned}\alpha\|T^nT_\alpha x-T_\alpha{}^2x\| &\le n(1-\alpha)^2\|x-T_\alpha x\|+\alpha\{(n-1)-n\alpha+\alpha^n\}\|x-T_\alpha x\|\\ &\quad +\alpha^{n+1}\|T_\alpha x-T_\alpha{}^2x\|\\ &= \{n-(n+1)\alpha+\alpha^{n+1}\}\|x-T_\alpha x\|+\alpha^{n+1}\|T_\alpha x-T_\alpha{}^2x\|\end{aligned}$$

as desired.

Thus we conclude that

$$\begin{aligned}\|T_\alpha x-T_\alpha{}^2x\| &\le \frac{(1-\alpha)a+(n-1)-n\alpha+\alpha^n}{1-\alpha^n}\|x-T_\alpha x\|\\ &= \{(a+n)(\textstyle\sum_{i=0}^{n-1}\alpha^i)^{-1}-1\}\|x-T_\alpha x\|\\ &= g(\alpha)\|x-T_\alpha x\|.\end{aligned}$$

Since T_α is nonexpansive, T_α satisfies all the hypothesis of Theorem 0. Thus for any $\varepsilon>0$, a point $u\in X$ satisfying $\|u-T_\alpha u\|\le(1-g(\alpha))\varepsilon$ is a fixed point of T_α or T_α has a fixed point in $\bar{B}(u,\varepsilon)\cap X\backslash B(u,s)$ where $s=\|u-T_\alpha u\|/2$. This implies the same conclusion for T, and completes our proof.

Note that Theorem 1 improves [4, Theorem 1] and [3, Theorem] for $k=1$.

THEOREM 2. *If* $T\in\Phi(n,a,k,X)$ *for some* $n\in\boldsymbol{N}$, $a\in[0,n)$ *and* $k>1$ *sufficiently close to* 1 *so that for any* $\alpha\in(0,1/k)$ *such that* $\tilde{g}(\alpha,k):=\dfrac{1-\alpha}{1-\alpha k}\left(a+\dfrac{k^n-1}{k-1}\right)(\sum_{i=0}^{n-1}(\alpha k)^i)^{-1}-1<1$, *then for any* $u\in X$ *and* $\varepsilon>0$ *satisfying*

$$\|u-T_\alpha x\|\le(1-\tilde{g}(\alpha,k))\varepsilon$$

either u *is a fixed point of* T *or there is a fixed point of* T *in* $\bar{B}(u,\varepsilon)\cap X\backslash B(u,s)$ *where* $s=\|u-T_\alpha u\|(1+k)^{-1}$.

Proof. As above, consider T_α defined by

$$T_\alpha x=(1-\alpha)x+\alpha TT_\alpha x.$$

Since T is k-lipschitzian, we have

$$\|T_\alpha x-T_\alpha{}^2x\|\le(1-\alpha)a\|x-T_\alpha x\|+\alpha k\|T^{n-1}T_\alpha x-T_\alpha{}^2x\|.$$

Using only the fact that T is k-lipschitzian we can also establish the following inequality:

$$\alpha k\|T^{m-1}T_\alpha x-T_\alpha{}^2x\|\le(1-\alpha)\left(\frac{k^m-1}{k-1}-\frac{1-\alpha^m k^m}{1-\alpha k}\right)\|x-T_\alpha x\| +\alpha^m k^m\|T_\alpha x-T_\alpha{}^2x\|.$$

For $m=2$,

$$\begin{aligned}\alpha k\|TT_\alpha x-T_\alpha{}^2x\|&\le\alpha k(1-\alpha)\|TT_\alpha x-T_\alpha x\|+\alpha^2k\|TT_\alpha x-TT_\alpha{}^2x\|\\&\le k(1-\alpha)^2\|x-T_\alpha x\|+\alpha^2k^2\|T_\alpha x-T_\alpha{}^2x\|\\&=(1-\alpha)\left(\frac{k^2-1}{k-1}-\frac{1-\alpha^2k^2}{1-\alpha k}\right)\|x-T_\alpha x\|\\&\quad+\alpha^2k^2\|T_\alpha x-T_\alpha{}^2x\|\end{aligned}$$

and for $m=n+1$,

$$\begin{aligned}\alpha k\|T^nT_\alpha x-T_\alpha{}^2x\|&\le\alpha k(1-\alpha)\|T^nT_\alpha x-T_\alpha x\|+\alpha^2k\|T^nT_\alpha x-TT_\alpha{}^2x\|\\&\le\alpha k(1-\alpha)(k^{n-1}+k^{n-2}+\cdots+k+1)\|TT_\alpha x-T_\alpha x\|\\&\quad+\alpha^2k^2\|T^{n-1}T_\alpha x-T_\alpha{}^2x\|\\&=k(1-\alpha)^2(k^{n-1}+k^{n-2}+\cdots+k+1)\|x-T_\alpha x\|\\&\quad+\alpha^2k^2\|T^{n-1}T_\alpha x-T_\alpha{}^2x\|\\&\le k(1-\alpha)^2\frac{k^n-1}{k-1}\|x-T_\alpha x\|\\&\quad+\alpha k(1-\alpha)\left(\frac{k^n-1}{k-1}-\frac{1-\alpha^nk^n}{1-\alpha k}\right)\|x-T_\alpha x\|\\&\quad+\alpha^{n+1}k^{n+1}\|T_\alpha x-T_\alpha{}^2x\|\\&=(1-\alpha)\left(\frac{k^{n+1}-1}{k-1}-\frac{1-\alpha^{n+1}k^{n+1}}{1-\alpha k}\right)\|x-T_\alpha x\|\\&\quad+\alpha^{n+1}k^{n+1}\|T_\alpha x-T_\alpha{}^2x\|.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}\|T_\alpha x-T_\alpha{}^2x\|&\le\frac{1-\alpha}{1-\alpha^nk^n}\left(\frac{k^n-1}{k-1}-\frac{1-\alpha^nk^n}{1-\alpha k}\right)\|x-T_\alpha x\|\\&=\frac{1-\alpha}{1-\alpha k}\left\{\frac{a+\frac{k^n-1}{k-1}}{\sum_{i=0}^{n-1}(\alpha k)^i}-1\right\}\|x-Tx\|.\end{aligned}$$

We put $\tilde{g}(\alpha, k)=\frac{1-\alpha}{1-\alpha k}\left\{\frac{a+\frac{k^n-1}{k-1}}{\sum_{i=0}^{n-1}(\alpha k)^i}-1\right\}$

then

$$\|T_\alpha x-T_\alpha{}^2x\|\leq\tilde{g}(\alpha, k)\|x-T_\alpha x\|.$$

Let us remark that $\tilde{g}(\alpha, 1)=g(\alpha)$ where $g(\alpha)$ is the function from the Theorem 1. Since $\tilde{g}(\alpha, 1)<1$ for $\alpha\in(\beta, 1)$ so for $k>1$ and sufficiently close to 1 there exists $\alpha\in(0, 1/k)$ such that $\tilde{g}(\alpha, k)<1$. Thus T satisfies all hypothesis of the Theorem 0, and hence the conclusion holds.

For fixed $n\in\boldsymbol{N}$, put

$\gamma_n(a)=\inf\{k$: there exist the set X and the map T such that $T\in\Phi(n, a, k, X)$ and Fix $T=\phi\}$.

We see that $\gamma_n:[0, n)\longrightarrow[1, \infty)$ and it is nonincreasing. The definition of $\gamma_n(a)$ implies that for arbitrary set X, if $T\in\Phi(n, a, k, X)$ and $k<\gamma_n(a)$, then T has at least one fixed point.

COROLLARY 1. [4, Theorem 2]. *For any $n\in\boldsymbol{N}$, and $a<n$, we have $\gamma_n(a)>1$.*

We shall now consider maps in the class $\Phi(2, a, k, X)$ for any $a<2$, $k>1$.

Let $T\in\Phi(2, a, k, X)$ and $x\in X$. For any $\alpha\in(0, 1)$ put

$$w=(1-\alpha)x+\alpha Tx$$
$$u=(1-\alpha)T^2x+\alpha Tx.$$

Then

$$\begin{aligned}\|w-Tw\|&\leq\|w-u\|+\|u-Tw\|\\&\leq(1-\alpha)\|x-T^2x\|+(1-\alpha)\|T^2x-Tx\|+\alpha\|Tx-Tw\|\\&\leq(1-\alpha)a\|x-Tx\|+(1-\alpha)k\|Tx-w\|+\alpha k\|x-w\|\\&=\{(1-\alpha)a+(1-\alpha)^2k+\alpha^2k\}\|x-Tx\|.\end{aligned}$$

Put $h(\alpha)=(1-\alpha)a+(1-\alpha)^2k+\alpha^2k$, then $h(\alpha)$ attains its minimum for $\alpha_0=(a+2k)/4k$ and $h(\alpha_0)=(4k^2+4ak-a^2)/8k$.

If $k<(2-a+\sqrt{(2-a)^2+a^2})/2$ then $h(\alpha_0)<1$.

Putting $S=(1-\alpha_0)I+\alpha_0T$, we obtain

$$\|S^2x-Sx\|\leq h(\alpha_0)\|Sx-x\|.$$

Since Fix $S=$Fix T, we have the following:

THEOREM 3. *If* $T\in\Phi(2, a, k, X)$ *for some* $a\in[0, 2)$ *and* $k\in(1, (2-a+\sqrt{(2-a)^2+a^2})/2)$ *then for any* $u\in X$ *and* $\varepsilon>0$ *satisfying* $\|u-Su\|\leq(1-h(\alpha_0))\varepsilon$ *either* u *is a fixed point of* T *or there is a fixed point of* T *in* $\bar{B}(u, \varepsilon)\cap X\backslash B(u, s)$ *where* $s=\alpha_0\|u-Tu\|(1+k)^{-1}$.

COROLLARY 2 [4, Theorem 3]. *In arbitrary Banach spaces, we have*

$$\gamma_2(a)\geq\frac{1}{2}(2-a+\sqrt{(2-a)^2+a^2}).$$

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