

ON CERTAIN LIPSCHITZIAN INVOLUTIONS IN BANACH SPACES

SEHIE PARK AND SANGSUK YIE

1. Introduction

In [3], [4], K. Goebel and E. Zlotkiewicz investigated conditions under which lipschitzian involutions or lipschitzian maps with nonexpansive square of a closed bounded convex subset X of a Banach space B have fixed points. A map $T : X \rightarrow X$ is called an involution if $T^2 = I$, where I denotes the identity map, and a k -lipschitzian if $\|Tx - Ty\| \leq k\|x - y\|$ holds for all $x, y \in X$. A 1-lipschitzian map is said to be nonexpansive.

In the present paper, the main results of [3], [4] are so strengthened that some information concerning the geometric estimations of fixed points are given.

Our tool in this paper is the following in [7], which is a consequence of the well-known variational principle of Ekeland [1], [2] for approximate solutions of minimization problems.

THEOREM 0. *Let V be a complete metric space and $f : V \rightarrow V$ be a map such that there exists an $L \in [0, 1)$ satisfying*

$$d(fx, f^2x) \leq Ld(x, fx) \text{ for any } x \in V.$$

If $F(x) = d(x, fx)$ on V is l. s. c., then

(1) $\lim f^n x = p$ exists for any $x \in V$,

$$d(f^n x, p) \leq \frac{L^n}{1-L} d(x, fx),$$

and p is a fixed point of f , and

(2) *for any $u \in V$ and $\varepsilon > 0$ satisfying*

$$F(u) \leq (1-L)\varepsilon,$$

f has a fixed point in $\bar{B}(u, \varepsilon)$. Further, if f is a quasi-lipschitzian with

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constant k , then either u is a fixed point of f or f has a fixed point in $\bar{B}(u, \varepsilon) \setminus B(u, s)$ where $s = d(u, fu)(1+k)^{-1}$.

Note that $\bar{B}(u, \varepsilon)$ denotes the closed ball with center u and radius ε , and $B(u, \varepsilon)$ the corresponding open ball.

A map $f: V \rightarrow V$ is called a quasi-lipschitzian with constant k if $\|fx - fp\| \leq k\|x - p\|$ holds for all $x \in V$ and for every fixed point p of f .

2. Main Results

The modulus of convexity of the space B is the function $\delta: [0, 2] \rightarrow [0, 1]$ defined by the following formula

$$\delta(\varepsilon) = \inf \{1 - \|\frac{x+y}{2}\| : x, y \in \bar{B}(0, 1), \|x-y\| \geq \varepsilon\}.$$

Note that the function $\delta(\varepsilon)$ is nonincreasing and convex.

Moreover, for any $x, y \in \bar{B}(0, r)$ and any a such that $0 \leq a \leq 2r$ and $\|x-y\| \geq a$, we have

$$\|\frac{x+y}{2}\| \leq (1 - \delta(\frac{a}{r}))r \quad [3].$$

Now we have our first result:

THEOREM 1. *Let X be a closed convex subset of a Banach space B and $T: X \rightarrow X$ a k -lipschitzian involution. If $L := k(1 - \delta(2/k))/2 < 1$, then for any $u \in X$ and $\varepsilon > 0$ satisfying*

$$\|u - Tu\| \leq (1-L)\varepsilon,$$

either u is a fixed point of T or there is a fixed point of T in $\bar{B}(u, \varepsilon/2) \cap X \setminus B(u, s)$ where $s = \|u - Tu\|(k+3)^{-1}$.

Proof. For any $x \in X$,

$$\begin{aligned} \|T\left(\frac{x+Tx}{2}\right) - x\| &= \|T\left(\frac{x+Tx}{2}\right) - T^2x\| \\ &\leq k\left\|\frac{x+Tx}{2} - Tx\right\| \\ &= \frac{k}{2}\|x - Tx\| \\ \|T\left(\frac{x+Tx}{2}\right) - Tx\| &\leq k\left\|\frac{x+Tx}{2} - x\right\| \\ &= \frac{k}{2}\|x - Tx\|. \end{aligned}$$

Thus, by the property of modulus of convexity, we have

$$\left\| \frac{x+Tx}{2} - T\left(\frac{x+Tx}{2}\right) \right\| \leq \left(1 - \delta\left(\frac{2}{k}\right)\right) \frac{k}{2} \|x - Tx\|.$$

Now if we put $G = \frac{1}{2}(I + T)$, then

$$\begin{aligned} \|Gx - G^2x\| &= \left\| \frac{Gx + TGx}{2} - Gx \right\| \\ &= \frac{1}{2} \|TGx - Gx\| \\ &\leq \frac{1}{2} \left(1 - \delta\left(\frac{2}{k}\right)\right) \frac{k}{2} \|x - Tx\| \\ &= \left(1 - \delta\left(\frac{2}{k}\right)\right) \frac{k}{2} \|x - Gx\| \\ &= L \|x - Gx\|. \end{aligned}$$

Therefore, by Theorem 0(1), $\lim G^n x = p$ exists for $x \in X$, and $p \in \text{Fix } G = \text{Fix } T$, the fixed point set. Since T is a k -lipschitzian, G is a $(k + 1)/2$ -lipschitzian and quasi-lipschitzian. Therefore, by Theorem 0(2), for any $u \in X$ with $\|u - Tu\| \leq (1 - L)\varepsilon$, we have $\|u - Gu\| = \|u - Tu\|/2 \leq (1 - L)\varepsilon/2$. Hence, u is a fixed point of G or there is a fixed point of G in $\bar{B}(u, \varepsilon/2) \cap X \setminus B(u, s)$ where $s = \|u - Gu\| / (1 + (k + 1)/2) = \|u - Tu\| / (k + 3)$. This completes our proof.

COROLLARY 1. [3, Theorem 1]. *Let X be a closed convex subset of a Banach space B and $T : X \rightarrow X$ a k -lipschitzian involution such that $k(1 - \delta(2/k))/2 < 1$. Then T has at least one fixed point.*

COROLLARY 2. *Let X be a closed convex subset of a Banach space B and $T : X \rightarrow X$ a k -lipschitzian involution. If $0 \leq k < 2$, then for any $u \in X$ and $\varepsilon > 0$ satisfying*

$$\|u - Tu\| \leq \left(1 - \frac{k}{2}\right) \varepsilon$$

the conclusion of Theorem 1 holds.

Proof. Let $L = k/2$ and $G = \frac{1}{2}(I + T)$. Then $L < 1$, and

$$\begin{aligned} \|Gx - G^2x\| &\leq \left(1 - \delta\left(\frac{2}{k}\right)\right) \frac{k}{2} \|x - Gx\| \\ &\leq \frac{k}{2} \|x - Gx\| \\ &= L \|x - Gx\|. \end{aligned}$$

Thus, by Theorem 1, we have the same conclusion to Theorem 1.

Corollary 2 improves [4, Theorem 1].

The characteristic of convexity of the space B is the number $\varepsilon_0 = \sup \{\varepsilon : \delta(\varepsilon) = 0\}$.

Some of Banach spaces can be fully characterized by the number ε_0 and the modulus of convexity. The following facts are known [3]

- (1) If $\varepsilon_0 < 1$, then B has normal structure,
- (2) B is uniformly non-square iff $\varepsilon_0 < 2$, and
- (3) B is strictly convex iff $\delta(2) = 1$.

THEOREM 2. *Let X be a closed convex bounded subset of a Banach space B with $\varepsilon_0 < 1$ and $\delta(2) = 1$, and $T : X \rightarrow X$ a k -lipschitzian map such that T^2 is nonexpansive. If $L := k(1 - \delta(2/k))/2 < 1$, then the conclusion of Theorem 1 holds.*

Proof. Since $\varepsilon_0 < 1$, B is uniformly non-square and in view of [5], it is reflexive, and moreover it has normal structure. Since T^2 is nonexpansive, by Kirk's fixed point theorem [6], the set $C^* = \{x : T^2x = x\}$ is nonempty. $\delta(2) = 1$ means the strict convexity of B and implies that C^* is convex. Obviously we have $T(C^*) = C^*$ and $T^2 = I$ on C^* . Hence, using Corollary 1 for the restriction of T on C^* , we can apply Theorem 1.

Theorem 2 improves [3, Theorem 2].

THEOREM 3. *Let X be a closed convex subset of a uniformly convex Banach space B and $T : X \rightarrow X$ a k -lipschitzian involution. If $L := k\delta^{-1}(1 - 1/k)/4 < 1$, then for any $u \in X$ and $\varepsilon > 0$ satisfying $\|u - Tu\| < (1 - L)\varepsilon$, either u is a fixed point of T or T has a fixed point in $\bar{B}(u, \varepsilon/2) \cap X \setminus B(u, s)$ where $s = \|u - Tu\|(k+3)^{-1}$.*

Proof. Let $G = (I + T)/2$ and let for $x \in X$, $y = Gx$ and $z = Ty$. Then

$$\begin{aligned} \|z - x\| &= \|Ty - x\| = \|Ty - T^2x\| \\ &\leq k\|y - Tx\| = k\|Gx - Tx\| \\ &= k\left\|\frac{x + Tx}{2} - Tx\right\| = \frac{k}{2}\|x - Tx\| \\ \|(2y - z) - x\| &= \|2Gx - Ty - x\| = \|x + Tx - Ty - x\| \\ &= \|Tx - Ty\| \leq k\|x - y\| = k\|x - Gx\| \\ &= k\left\|x - \frac{x + Tx}{2}\right\| = \frac{k}{2}\|x - Tx\| \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{z + (2y - z)}{2} - x \right\| &= \|y - x\| = \|x - Gx\| \\ &= \frac{1}{2} \|x - Tx\|. \end{aligned}$$

Thus, by the property of modulus of convexity, we have

$$\|z - (2y - z)\| \leq \frac{k}{2} \delta^{-1} \left(1 - \frac{1}{k}\right) \|x - Tx\|.$$

But since

$$\begin{aligned} \|z - (2y - z)\| &= 2\|y - z\| = 2\|Gx - Ty\| \\ &= 4\|Gx - G^2x\| \end{aligned}$$

and

$$\|x - Tx\| = 2\|x - Gx\|,$$

we have

$$4\|Gx - G^2x\| \leq k\delta^{-1} \left(1 - \frac{1}{k}\right) \|x - Tx\|$$

i. e. ,

$$\begin{aligned} \|Gx - G^2x\| &\leq \frac{k}{4} \delta^{-1} \left(1 - \frac{1}{k}\right) \|x - Gx\| \\ &= L\|x - Gx\|, \end{aligned}$$

and

$$\|x - Gx\| \leq (1 - L)\varepsilon/2.$$

Thus G satisfies all the hypothesis of Theorem 0, and we conclude our result.

Theorem 3 strengthens [4, Theorem 2].

THEOREM 4. *Let X be a closed bounded convex subset of a uniformly convex Banach space B , and $T : X \rightarrow X$ a k -lipschitzian map such that T^2 is nonexpansive. If $L = k\delta^{-1}(1 - 1/k)/4 < 1$, then the conclusion of Theorem 3 holds.*

Proof. Since every uniformly convex Banach space is strictly convex, reflexive and has normal structure, by Theorem 2 and Theorem 3, we obtain our result.

Theorem 4 strengthens [4, Theorem 3].

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Seoul National University
Seoul 151, Korea
and
Soong Jun University
Seoul 151, Korea